Discussion 10B

1. **Gram-Schmidt Algorithm**

   Let's apply Gram-Schmidt orthonormalization to a list of three linearly independent vectors $[\vec{s}_1, \vec{s}_2, \vec{s}_3]$.

   (a) Let's say we had two collections of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\{\vec{w}_1, \ldots, \vec{w}_n\}$. **How can we prove that** $\mathrm{Span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = \mathrm{Span}(\{\vec{w}_1, \ldots, \vec{w}_n\})$?

   (b) **Find unit vector $\vec{q}_1$ such that** $\mathrm{Span}(\{\vec{q}_1\}) = \mathrm{Span}(\{\vec{s}_1\})$**, where $\vec{s}_1$ is nonzero.**

   (c) Let's say that we wanted to write

   $$\vec{s}_2 = c_1 \vec{q}_1 + \vec{z}_2 \tag{1}$$

   where $c_1 \vec{q}_1$ entirely represents the component of $\vec{s}_2$ in the direction of $\vec{q}_1$, and $\vec{z}_2$ represents the component of $\vec{s}_2$ that is distinctly *not* in the direction of $\vec{q}_1$ (i.e. $\vec{z}_2$ and $\vec{q}_1$ are orthogonal).

   Given $\vec{q}_1$ from the previous step, **find c_1 as in eq. (1), and use $\vec{z}_2$ to find unit vector $\vec{q}_2$ such that** $\mathrm{Span}(\{\vec{q}_1, \vec{q}_2\}) = \mathrm{Span}(\{\vec{s}_1, \vec{s}_2\})$ **and $\vec{q}_2$ is orthogonal to $\vec{q}_1$. Show that** $\mathrm{Span}(\{\vec{q}_1, \vec{q}_2\}) = \mathrm{Span}(\{\vec{s}_1, \vec{s}_2\})$**.**

(d) **What would happen if $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ were *not* linearly independent, but rather $\vec{s}_1$ were a multiple of $\vec{s}_2$?**

(e) Now given $\vec{q}_1$ and $\vec{q}_2$ in parts **1.**b and **1.**c, **find $\vec{q}_3$ such that** $\text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$**, and $\vec{q}_3$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$, and finally $\|\vec{q}_3\| = 1$.** You do not have to show that the two spans are equal.

Contributors:
- Anish Muthali.
- Regina Eckert.
- Druv Pai.
- Neelesh Ramachandran.