1. Uncontrollability

Recall that, for a $n$-dimensional, discrete-time linear system to be controllable, we require that the controllability matrix $C = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \ldots & AB & B \end{bmatrix}$ to be rank $n$.

Consider the following discrete-time system with the given initial state:

$$\vec{x}[i + 1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i]$$  \hspace{1cm} (1)

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (2)

(a) **Is the system controllable?**

**Solution:**

$$C = \begin{bmatrix} A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$  \hspace{1cm} (3)

Since the controllability matrix $C$ only has rank 2, the system is not controllable. We would need it to be rank 3 here to span the full space $\mathbb{R}^3$.

(b) **Show that we can write the $i$th state as**

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ -3x_1[i - 1] + x_3[i - 1] \\ x_2[i - 1] + 2u[i - 1] \end{bmatrix}$$  \hspace{1cm} (4)

**Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some $\ell$? If so, for what input sequence $u[i]$ up to $i = \ell - 1$?**

**Solution:** We can write:

$$\vec{x}[i] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i - 1] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i - 1]$$  \hspace{1cm} (5)

$$= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[i - 1] \\ x_2[i - 1] \\ x_3[i - 1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i - 1]$$  \hspace{1cm} (6)

$$= \begin{bmatrix} 2x_1[i - 1] \\ -3x_1[i - 1] + x_3[i - 1] \\ x_2[i - 1] + 2u[i - 1] \end{bmatrix}$$  \hspace{1cm} (7)
\[
\begin{bmatrix}
2^i x_1[0] \\
-3x_1[i-1] + x_3[i-1] \\
x_2[i-1] + 2u[i-1]
\end{bmatrix} = \begin{bmatrix}
-3x_1[i-1] + x_3[i-1] \\
x_2[i-1] + 2u[i-1]
\end{bmatrix}
\]

Note that in this expression for $\vec{x}[i]$, $x_1[i] = 2^i$ is decoupled from all other states and inputs. From this expression we also see that there is no choice of inputs us to get to $x_1[\ell] = -2$. Therefore, we will never be able to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for any $\ell$.

c) Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some $\ell$? For what input sequence $u[i]$ for $i = 0$ to $i = \ell - 1$?

HINT: Use the result for $\vec{x}[i]$ from the previous part.

Solution: We need $\ell = 1$ since $x_1[i] = 2^i x_1[0] = 2^i$. Hence,

\[
\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}
\]

We realize that the first two entries of $\vec{x}[1]$ are exactly what we want. Thus, we have to choose $u[0]$ so the third entry is $-2$. If we choose $u[0] = -1$, then we have reached our desired state.

Thus we see that a system being uncontrollable does not mean we are unable to reach anything at all, but just that the range that can be reached is limited.

d) Find the set of all $\vec{x}[2]$, given that you are free to choose any $u[0]$ and $u[1]$.

Solution:

\[
\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}
\]

\[
\vec{x}[2] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} u[1] \\ \end{bmatrix}
\]

\[
= \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix}
\]

Since we can set $u[0]$ and $u[1]$ arbitrarily, we can reach any state of the form $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span} \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ after two timesteps. This means that we can reach any value for the second and third components of $\vec{x}[2]$; contrast this with how the first component of the state vector is fixed at 4 after two timesteps, and cannot be changed by the inputs.
Alternative Solution:

Notice that we can write

\[ \vec{x}[2] = A^2 \vec{x}[0] + ABu[0] + Bu[1] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} \]  

(14)

Hence, \( \vec{x}[2] \) will be \( \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \) plus whatever is in the column space of

\[ \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \]  

(15)

This gives the same answer as before, i.e. that

\[ \vec{x}[2] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \vec{p} \]  

(16)

where \( \vec{p} \in \text{span} \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \).

Any desired \( x_2[i] \) and \( x_3[i] \) that we can possibly reach can be obtained in only two or fewer timesteps. Hence, every reachable state can be written as

\[ \vec{x}[i] = \begin{bmatrix} 2^i \\ 0 \\ 0 \end{bmatrix} + \vec{p} \]  

(17)

with \( \vec{p} \) defined as above. This will also tell us why the desired goal in part 1.b is unreachable.

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