Discussion 10A

The following notes are useful for this discussion: Note 13, Note 15.

1. 2×2 Upper Triangularization Example

Solution: In lecture, we have been motivated by the goal of getting to a coordinate system in which the eigenvalues are on the diagonal, and there are only zeros below the diagonal. There can be "stuff" (nonzero entries) above the diagonal. When this is done to the A matrix representing a time-evolving system, we can view the system as a cascade of scalar systems — with each one potentially being an input to the ones that come "above" it.

Previously in this course, we have seen the value of changing our coordinates to be eigenbasis-aligned, because we can then view the system as a set of parallel scalar systems. Diagonalization causes these scalar equations to be fully uncoupled such that they can be solved separately. But even when we cannot diagonalize, we can *upper-triangularize* such that we can still solve the equations one at a time, from the "bottom up".

Recall that Schur Decomposition is a method by which we can take some M matrix and decompose it into $U^\top T U$ where U is an orthonormal matrix and T is an upper triangular matrix. This is the Real Schur Decomposition algorithm from Note 15 for reference.

Algorithm 1 Real Schur Decomposition

Require: A square matrix $A \in \mathbb{R}^{n\times n}$ with real eigenvalues.

Ensure: An orthonormal matrix $U \in \mathbb{R}^{n\times n}$ and an upper-triangular matrix $T \in \mathbb{R}^{n\times n}$ such that $A = UTU^\top$.

```
function REALSCHURDECOMPOSITION(A)
    if A is 1 × 1 then
        return [1], A
    end if
    (q⃗_1, λ_1) := FINDEIGENVECTOREIGENVALUE(A)
    Q := EXTENDBASIS({q⃗_1}, ℝ^n)    ▷ Extend {q⃗_1} to a basis of ℝ^n using Gram-Schmidt; see Note 13
    Unpack Q := [q⃗_1  Q̃]
    Compute and unpack Q^⊤ A Q = [ λ_1  ã⃗_12^⊤ ; 0⃗_{n−1}  Ã_22 ]
    (P, T̃) := REALSCHURDECOMPOSITION(Ã_22)
    U := [q⃗_1  Q̃P]
    T := [ λ_1  ã⃗_12^⊤ P ; 0⃗_{n−1}  T̃ ]
    return (U, T)
end function
```

In this problem, we are going to be working with the following 2×2 matrix:

$$A = \begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} \tag{1}$$

(a) Remember that diagonalization is another tool we have learned that can be used to decompose a matrix. However, there is the restriction that our transformation V (which was chosen to be a matrix of the eigenvectors of the A matrix) must be invertible. That means that we needed n linearly independent eigenvectors for a matrix $A \in \mathbb{R}^{n \times n}$. **For the given matrix A, calculate its eigenvalues and eigenvectors and determine whether or not we can diagonalize the matrix.**

Solution: To solve for eigenvalues we take

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda + 8 & 5 \\ -5 & \lambda - 2 \end{bmatrix}\right) \tag{2}$$

$$(\lambda + 8)(\lambda - 2) - (-5)(5) = 0 \tag{3}$$

$$\lambda^2 + 8\lambda - 2\lambda - 16 + 25 = 0 \tag{4}$$

$$\lambda^2 + 6\lambda + 9 = 0 \tag{5}$$

$$(\lambda + 3)^2 = 0 \tag{6}$$

Thus, we have $\lambda_1 = \lambda_2 = -3$. Now, let's solve for the eigenvectors corresponding to our eigenvalues.

$$\begin{bmatrix} -3 + 8 & 5 \\ -5 & -3 - 2 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

$$\vec{v}_1 = \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \beta \in \mathbb{R} \tag{9}$$

The calculation for $\vec{v}_2$ would be the same since $\lambda_1 = \lambda_2$. However, if $\vec{v}_1 = \vec{v}_2$, then we have linearly dependent eigenvectors and our transformation matrix for diagonalization would be:

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \tag{10}$$

which is clearly not invertible. Therefore, we cannot diagonalize our system and must use other means of solving this system.

(b) Hopefully, in the previous part you observed that this matrix has repeated eigenvalues and as a result did not have linearly independent eigenvectors. Instead let's try and upper triangularize the system. Recall that the first step of Schur Decomposition is to calculate an eigenvalue, eigenvector pair. We have already done that in **1.a**, so we can directly use our calculations.

Using Gram-Schmidt, extend an orthogonal basis for $\mathbb{R}^2$ from our eigenvector $\vec{v}_1$. In other words, find an orthonormal set of vectors $Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$ where $\text{Span}(\vec{q}_1, \vec{q}_2) = \mathbb{R}^2$ *(HINT: What vectors do we typically append for basis extension?)*

Solution: When extending a basis via Gram Schmidt, we need to append an arbitrary basis, which we usually choose to be the standard basis vectors for $\mathbb{R}^2$, which are the columns of $I_{2 \times 2}$

or $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. Now, we are running Gram-Schmidt on the set $\{\vec{v}_1, \vec{e}_1, \vec{e}_2\} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$.

The first step is to calculate $\vec{q}_1$. We will take our first vector $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and ensure that it is normalized, since we are forming an orthonormal basis. This yields:

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \tag{11}$$

Next we will calculate $\vec{q}_2$. Recall to calculate an orthogonal vector we wish to decompose our next vector $\vec{s}_2$ into the component in the direction of $\vec{q}_1$ and the component orthogonal to $\vec{q}_1$. The component of $\vec{s}_2$ that is in the direction of $\vec{q}_1$ is by definition of the projection of $\vec{s}_2$ onto $\vec{q}_1$. Thus we can say:

$$\vec{s}_2 = \text{proj}_{\vec{q}_1}(\vec{s}_2) + \vec{z}_2 \tag{12}$$

where $\vec{z}_2$ is the component of $\vec{s}_2$ orthogonal to $\vec{q}_1$. We want $\vec{z}_2$, so we can rearrange our equation to be:

$$\vec{z}_2 = \vec{s}_2 - \text{proj}_{\vec{q}_1}(\vec{s}_2) \tag{13}$$

$$\vec{z}_2 = \vec{s}_2 - \frac{\langle \vec{s}_2, \vec{q}_1 \rangle}{\langle \vec{q}_1, \vec{q}_1 \rangle}\vec{q}_1 \tag{14}$$

$$\vec{z}_2 = \vec{s}_2 - (\vec{q}_1^\top \vec{s}_2)\vec{q}_1 \tag{15}$$

We make the last simplication from the fact that $\vec{q}_1$ is an orthonormal vector and therefore $\langle \vec{q}_1, \vec{q}_1 \rangle = \|\vec{q}_1\|^2 = 1$ Using the fact that $\vec{s}_2 = \vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, we can solve for $\vec{z}_2$:

$$\vec{z}_2 = \begin{bmatrix}1\\0\end{bmatrix} - \left(\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \tag{16}$$

$$\vec{z}_2 = \begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \tag{17}$$

$$\vec{z}_2 = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} \tag{18}$$

We have now found the component of $\vec{e}_1$ that is orthogonal to $\vec{q}_1$. The last step is to normal our vector:

$$\vec{q}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \tag{19}$$

Putting together $\vec{q}_1$ and $\vec{q}_2$, we have found an orthonormal basis:

$$Q = \begin{bmatrix}\vec{q}_1 & \vec{q}_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} \tag{20}$$

(c) Now that we have calculated some $Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$, let's apply this transformation to our original matrix A. **Calculate $Q^\top AQ$ and comment on the resulting matrix.**

Solution:

$$Q^\top AQ = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^\top \begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{2}} & -\frac{13}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & -\frac{7}{\sqrt{2}} \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{2}} & -\frac{13}{\sqrt{2}} \\ -\frac{3}{\sqrt{2}} & \frac{7}{\sqrt{2}} \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} -3 & 10 \\ 0 & -3 \end{bmatrix} \tag{24}$$

We have successfully upper triangularized a 2×2 matrix. Notice that we only had to execute one iteration of the Schur Decomposition algorithm. With larger matrices you will have to repeat this process recursively. In homework, you will get practice with upper triangularizing a 3×3 matrix!

(d) **How do the eigenvalues of the original A matrix connect to the upper triangular matrix $T = Q^\top AQ$ that we calculated in the previous part.**

Solution: Notice that the eigenvalues $\lambda_1 = \lambda_2 = -3$ are along the diagonal of the upper triangular matrix. This is a nice property of upper triangular matrices.

(e) Let's say you were given a system:

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = \begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} \vec{x}(t) \tag{25}$$

Describe how you could solve for $\vec{x}(t)$ given an initial condition $\vec{x}(0)$.

Solution: We can convert the system into an upper triangular system using the transformation that we just solved for where $\vec{x}(t) = Q\vec{\tilde{x}}(t)$.

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = A\vec{x}(t) \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Q\vec{\tilde{x}}(t)\right) = AQ\vec{\tilde{x}}(t) \tag{27}$$

$$Q\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = AQ\vec{\tilde{x}}(t) \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = Q^\top AQ\vec{\tilde{x}}(t) \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = T\vec{\tilde{x}}(t) \tag{30}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = \begin{bmatrix} -3 & 10 \\ 0 & -3 \end{bmatrix}\vec{\tilde{x}}(t) \tag{31}$$

Notice that the second row of our system is uncoupled and therefore we can treat it as a scalar equation as solve for it.

$$\frac{\mathrm{d}\tilde{x}_2(t)}{\mathrm{d}t} = -3\tilde{x}_2(t) \tag{32}$$

Once we've found a solution for $\tilde{x}_2(t)$, when we can use this to help solve for $\tilde{x}_1(t)$ by substituting in our solution. From there we will find a solution for $\vec{\tilde{x}}(t)$. Lastly, we need to convert back into our original coordinates, so apply $\vec{x}(t) = Q\vec{\tilde{x}}(t)$.

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