1. Towards Upper-Triangulation By An Orthonormal Basis

Previously in this course, we have seen the value of changing our coordinates to be eigenbasis-aligned, because we can then view the system as a cascade of scalar systems. If we have a diagonalization, then these scalar equations are fully uncoupled, and can therefore be treated completely separately. But even when we cannot diagonalize, we can upper-triangularize in a way that allows us to solve the equations one at a time, from the "bottom up".

In this problem, to better understand the steps involved, we will use the following concrete example:

\[
S_{[3 \times 3]} = \begin{bmatrix}
5 & 5 & 1 \\
-1 & 3 & 16 \\
-6 & 1 & 23 \\
\end{bmatrix}
\]  

and figure out the general case by abstracting variables. Note that there is a datahub link to a jupyter notebook on the website, which will allow you to perform the numerical calculations quickly to connect the symbolic analysis to an example, but without being time-consuming in the process.\(^1\)

(a) Consider a non-zero vector \( \vec{u}_0 \in \mathbb{R}^n \). Can you think of a way to extend it to a set of basis vectors for \( \mathbb{R}^n \)? In other words, find \( \vec{u}_1, \ldots, \vec{u}_{n-1} \), such that \( \text{span}(\vec{u}_0, \vec{u}_1, \ldots, \vec{u}_{n-1}) = \mathbb{R}^n \). To begin with, consider \( \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \). Can you get an orthonormal basis from what you just constructed?

*Hint: what was the last discussion all about? Also, the given vector isn’t normalized yet!*

\(^1\)This particular matrix has an additional special property of symmetry, but we won’t be invoking that here.
(b) Now consider a real eigenvalue $\lambda_1$, and the corresponding (normalized) eigenvector $\vec{v}_1 \in \mathbb{R}^n$ of a square matrix $M \in \mathbb{R}^{n \times n}$. From the previous part, we know that we can extend $\vec{v}_1$ to an orthonormal basis of $\mathbb{R}^n$. We will denote the basis by

$$U = \begin{bmatrix} | & | & \cdots & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}$$

where $\vec{u}_1 = \vec{v}_1$ (note that this eigenvector is already normalized).

Our goal is to look at what the matrix $M$ looks like in the coordinate system defined by the basis $U$.

Compute $U^\top MU$ by writing $U = [\vec{v}_1 \ R],$ where $R \equiv \begin{bmatrix} | & | & \cdots & | \\ \vec{r}_1 & \vec{r}_2 & \cdots & \vec{r}_{n-1} \end{bmatrix}$.

(c) Show that $U^{-1} = U^\top$.
(d) Define \( Q = R^\top MR \). Look at the first column and the first row of \( U^\top MU \) and show that:

\[
M = U \begin{bmatrix}
\lambda_1 & \bar{a}^\top \\
0 & Q
\end{bmatrix} U^\top
\]

Here, \( \bar{a} \) is a symbolic vector related to \( M, R, \) and \( \bar{v}_1 \) (we will show the relation!).
(e) Now, we can recurse on $Q$ to get:

$$Q = \begin{bmatrix} \vec{v}_2 & Y \end{bmatrix} \begin{bmatrix} \lambda_2 & \vec{b}^\top \\ 0 & P \end{bmatrix} \begin{bmatrix} \vec{v}_2 & Y \end{bmatrix}^\top$$

where we have taken $\vec{v}_2 \in \mathbb{R}^{n-1}$, a normalized eigenvector of $Q$, associated with eigenvalue $\lambda_2$. Again $\vec{v}_2$ is extended into an orthonormal basis to form $[\vec{v}_2 \ Y]$.

Plug this form of $Q$ into $M$ above, to show that:

$$M = \begin{bmatrix} \vec{v}_1 & R\vec{v}_2 & RY \end{bmatrix} \begin{bmatrix} \lambda_1 & \vec{a}_1 & \vec{\tilde{a}}_\text{rest} \\ 0 & \lambda_2 & \vec{b}^\top \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} \vec{v}_1 & R\vec{v}_2 & RY \end{bmatrix}^\top$$

where we define $\vec{\tilde{a}}$ to be the "adjusted" $\vec{a}$ to account for the substitution of $Q$; $\vec{\tilde{a}}^\top = \vec{\tilde{a}}^\top [\vec{v}_2 \ Y]$. 

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