Discussion 9B

The following notes are useful for this discussion: Note 13.

1. Gram-Schmidt Algorithm

Let's apply Gram-Schmidt orthonormalization to a list of three linearly independent vectors $[\vec{s}_1, \vec{s}_2, \vec{s}_3]$.

(a) Let's say we had two collections of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\{\vec{w}_1, \ldots, \vec{w}_n\}$. **How can we prove that** $\text{Span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = \text{Span}(\{\vec{w}_1, \ldots, \vec{w}_n\})$**?**

Solution: Notice that taking the span of some vectors gives you a set of vectors. So, when proving two sets S_1 and S_2 are equal, we can show that $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$. We can show $S_1 \subseteq S_2$ by showing that, if $a \in S_1$, then $a \in S_2$. Likewise, we can show $S_2 \subseteq S_1$ by showing that, if $b \in S_2$, then $b \in S_1$.

In the context of the given problem, we can show that $\text{Span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) = \text{Span}(\{\vec{w}_1, \ldots, \vec{w}_n\})$ by first showing $\text{Span}(\{\vec{v}_1, \ldots, \vec{v}_n\}) \subseteq \text{Span}(\{\vec{w}_1, \ldots, \vec{w}_n\})$. That is, we can show that $\vec{v}_i \in \text{Span}(\{\vec{w}_1, \ldots, \vec{w}_n\})$ for every $i = 1$ to $i = n$. Next, we can show $\text{Span}(\{\vec{w}_1, \ldots, \vec{w}_n\}) \subseteq \text{Span}(\{\vec{v}_1, \ldots, \vec{v}_n\})$ by showing that $\vec{w}_i \in \text{Span}(\{\vec{v}_1, \ldots, \vec{v}_n\})$ for every $i = 1$ to $i = n$.

(b) **Find unit vector** $\vec{q}_1$ **such that** $\text{Span}(\{\vec{q}_1\}) = \text{Span}(\{\vec{s}_1\})$**, where** $\vec{s}_1$ **is nonzero.**

Solution: Note that any $\vec{v} \in \text{Span}(\{\vec{s}_1\})$ can be written as $\vec{v} = a\vec{s}_1$ for some $a \in \mathbb{R}$. We need $\vec{q}_1 \in \text{Span}(\{\vec{s}_1\})$ and we need it to be a unit vector. Hence, we can write

$$\vec{q}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|}. \tag{1}$$

Next, we need to show $\vec{s}_1 \in \text{Span}(\{\vec{q}_1\})$. We can see that $\vec{s}_1 = a\vec{q}_1$ where $a = \|\vec{s}_1\|$.

(c) Let's say that we wanted to write

$$\vec{s}_2 = c_1\vec{q}_1 + \vec{z}_2 \tag{2}$$

where $c_1\vec{q}_1$ entirely represents the component of $\vec{s}_2$ in the direction of $\vec{q}_1$, and $\vec{z}_2$ represents the component of $\vec{s}_2$ that is distinctly *not* in the direction of $\vec{q}_1$ (i.e. $\vec{z}_2$ and $\vec{q}_1$ are orthogonal).

Given $\vec{q}_1$ from the previous step, **find** c_1 **as in eq. (2), and use** $\vec{z}_2$ **to find unit vector** $\vec{q}_2$ **such that** $\text{Span}(\{\vec{q}_1, \vec{q}_2\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2\})$ **and** $\vec{q}_2$ **is orthogonal to** $\vec{q}_1$**. Show that** $\text{Span}(\{\vec{q}_1, \vec{q}_2\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2\})$**.**

Solution: To find c_1, we can compute the projection of $\vec{s}_2$ onto $\vec{q}_1$, namely

$$\text{proj}_{\vec{q}_1}(\vec{s}_2) = \frac{\vec{q}_1^\top \vec{s}_2}{\left(\underbrace{\vec{q}_1^\top \vec{q}_1}_{1}\right)}\vec{q}_1 = \underbrace{\left(\vec{q}_1^\top \vec{s}_2\right)}_{c_1}\vec{q}_1 \tag{3}$$

This projection represents all the components of $\vec{s}_2$ that are in the direction of $\vec{q}_1$. To find $\vec{z}_2$, we can use eq. (2) to obtain

$$\vec{z}_2 = \vec{s}_2 - \left(\vec{q}_1^\top \vec{s}_2\right)\vec{q}_1 \tag{4}$$

which, by design, is orthogonal to $\vec{q}_1$ since it has no components in the direction of $\vec{q}_1$. We have satisfied the orthogonality condition with $\vec{z}_2$, so all that is left is to normalize this quantity to find $\vec{q}_2$:

$$\vec{q}_2 = \frac{\vec{z}_2}{\|\vec{z}_2\|} \tag{5}$$

Next, we need to show the two spans are equal. First, we can show $\text{Span}(\{\vec{q}_1, \vec{q}_2\}) \subseteq \text{Span}(\{\vec{s}_1, \vec{s}_2\})$. From part **1.**b, we already know $\vec{q}_1 \in \text{Span}(\{\vec{s}_1\}) \subseteq \text{Span}(\{\vec{s}_1, \vec{s}_2\})$. We can rewrite $\vec{q}_2$ as

$$\vec{q}_2 = \alpha\vec{s}_2 + \beta\vec{q}_1 \tag{6}$$

for $\alpha = \frac{1}{\|\vec{z}_2\|}$ and $\beta = \frac{-(\vec{q}_1^\top \vec{s}_2)}{\|\vec{z}_2\|}$. We know $\vec{q}_1 = a\vec{s}_1$ for $a = \frac{1}{\|\vec{s}_1\|}$ (from part **1.**b), so we can write

$$\vec{q}_2 = \alpha\vec{s}_2 + a\beta\vec{s}_1 \tag{7}$$

so $\vec{q}_2 \in \text{Span}(\{\vec{s}_1, \vec{s}_2\})$.

Next, we can show $\text{Span}(\{\vec{s}_1, \vec{s}_2\}) \subseteq \text{Span}(\{\vec{q}_1, \vec{q}_2\})$. From the **1.**b, we know $\vec{s}_1 \in \text{Span}(\{\vec{q}_1\}) \subseteq \text{Span}(\{\vec{q}_1, \vec{q}_2\})$. Now, we can perform algebraic manipulation and rewrite eq. (6) to say

$$\vec{s}_2 = \frac{\vec{q}_2}{\alpha} - \frac{\beta\vec{q}_1}{\alpha} \tag{8}$$

so $\vec{s}_2 \in \text{Span}(\{\vec{q}_1, \vec{q}_2\})$. Hence, we have shown that $\text{Span}(\{\vec{q}_1, \vec{q}_2\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2\})$.

Intuitive Explanation on Projections for Orthogonalization:

The idea behind why we take projections and calculate projection error can be seen as a method to extract $\vec{z}_2$ from

$$\vec{s}_2 = c_1\vec{q}_1 + \vec{z}_2 \tag{9}$$

where we choose this decomposition of $\vec{s}_2$ such that $c_1\vec{q}_1$ and $\vec{z}_2$ are orthogonal. That is, we will use the term $c_1\vec{q}_1$ to represent the component of $\vec{s}_2$ in the direction of $\vec{q}_1$, and $\vec{z}_2$ to represent the component of $\vec{s}_2$ that is distinctly *not* in the direction of $\vec{q}_1$. We can solve for c_1 using projections. By subtracting this part out as in eq. (4), we are left with a vector $\vec{z}_2$ that does not have any components in the direction of $\vec{q}_1$. Hence, it will be orthogonal to $\vec{q}_1$. See fig. 1 for an intuitive plot of what this decomposition could look like.

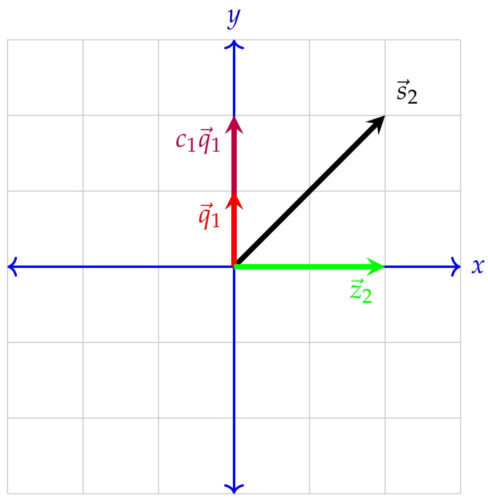

Figure 1: Decomposition of $\vec{s}_2$

(d) **What would happen if $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ were *not* linearly independent, but rather $\vec{s}_1$ were a multiple of $\vec{s}_2$?**

Solution: If $\vec{s}_2$ is a multiple of $\vec{s}_1$, then $\vec{z}_2 = 0$. This means that the projection of $\vec{s}_2$ onto $\text{Span}(\{\vec{s}_1\})$ is just $\vec{s}_2$, so we have found an orthonormal basis for $\text{Span}(\{\vec{s}_1, \vec{s}_2\})$, in particular the basis $\{\vec{q}_1\}$. Hence, we can move onto $\vec{s}_3$ and continue the algorithm from there.

(e) Now given $\vec{q}_1$ and $\vec{q}_2$ in parts **1.b** and **1.c**, **find $\vec{q}_3$ such that** $\text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$**, and $\vec{q}_3$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$, and finally $\|\vec{q}_3\| = 1$.** You do not have to show that the two spans are equal.

Solution: Based on the intuitive explanation from part **1.c**, we would like to write

$$\vec{s}_3 = c_1\vec{q}_1 + c_2\vec{q}_2 + \vec{z}_3 \tag{10}$$

where $c_1\vec{q}_1$ represents the component of $\vec{s}_3$ that is in the direction of only $\vec{q}_1$, $c_2\vec{q}_2$ represents the component that is in the direction of only $\vec{q}_2$, and $\vec{z}_3$ represents the component that is distinctly *not* in the directions of $\vec{q}_1$ and $\vec{q}_2$. Note that $\vec{q}_1$ and $\vec{q}_2$ are in distinctly different directions, since they are orthogonal (this allows us to claim that $c_1\vec{q}_1$ and $c_2\vec{q}_2$ represent distinctly different directional components of $\vec{s}_3$).

We can compute c_1 and c_2 by projections. Namely,

$$c_1\vec{q}_1 = \text{proj}_{\vec{q}_1}(\vec{s}_3) = \frac{\vec{q}_1^\top \vec{s}_3}{\|\vec{q}_1\|^2}\vec{q}_1 = \underbrace{\left(\vec{q}_1^\top \vec{s}_3\right)}_{c_1}\vec{q}_1 \tag{11}$$

$$c_2\vec{q}_2 = \text{proj}_{\vec{q}_2}(\vec{s}_3) = \frac{\vec{q}_2^\top \vec{s}_3}{\|\vec{q}_2\|^2}\vec{q}_2 = \underbrace{\left(\vec{q}_2^\top \vec{s}_3\right)}_{c_2}\vec{q}_2 \tag{12}$$

To find $\vec{z}_3$, we can subtract out $c_1\vec{q}_1$ and $c_2\vec{q}_2$, namely:

$$\vec{z}_3 = \vec{s}_3 - \left(\vec{q}_1^\top \vec{s}_3\right)\vec{q}_1 - \left(\vec{q}_2^\top \vec{s}_3\right)\vec{q}_2 \tag{13}$$

All that is left is to normalize this quantity, that is

$$\vec{q}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|} \tag{14}$$

(f) **(PRACTICE) Confirm that** $\text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) = \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$**.**

Solution: We already showed that $\vec{q}_1, \vec{q}_2 \in \text{Span}(\{\vec{s}_1, \vec{s}_2\}) \subseteq \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$ and also $\vec{s}_1, \vec{s}_2 \in \text{Span}(\{\vec{q}_1, \vec{q}_2\}) \subseteq \text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\})$. It remains to show that $\vec{q}_3 \in \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$ (so we can show $\text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}) \subseteq \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$) and that $\vec{s}_3 \in \text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\})$ (so we can show $\text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}) \subseteq \text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\})$)

To show $\vec{q}_3 \in \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$:

$$\vec{q}_3 = \frac{\vec{z}_3}{\|\vec{z}_3\|} = \underbrace{\gamma}_{\frac{1}{\|\vec{z}_3\|}} \left(\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 - \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2\right) \tag{15}$$

$$= \gamma \left(\vec{s}_3 - \left(\vec{s}_3^\top \vec{q}_1\right) \underbrace{\vec{q}_1}_{a\vec{s}_1} - \left(\vec{s}_3^\top \vec{q}_2\right) \underbrace{\vec{q}_2}_{\alpha\vec{s}_2 + a\beta\vec{s}_1}\right) \tag{16}$$

$$= \gamma\vec{s}_3 + \left(-\alpha\left(\vec{s}_3^\top \vec{q}_2\right)\right)\vec{s}_2 + \left(-a\left(\vec{s}_3^\top \vec{q}_1\right) - a\beta\left(\vec{s}_3^\top \vec{q}_2\right)\right)\vec{s}_1 \tag{17}$$

where $a = \frac{1}{\|\vec{s}_1\|}$, $\alpha = \frac{1}{\|\vec{z}_2\|}$, and $\beta = \frac{-(\vec{q}_1^\top \vec{s}_2)}{\|\vec{z}_2\|}$ (taken from eq. (7)). So, $\vec{q}_3 \in \text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\})$. Now, to show $\vec{s}_3 \in \text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\})$, we can perform algebraic manipulation on eq. (16):

$$\vec{s}_3 = \frac{1}{\gamma}\left(\vec{q}_3 + \left(\vec{s}_3^\top \vec{q}_1\right)\vec{q}_1 + \left(\vec{s}_3^\top \vec{q}_2\right)\vec{q}_2\right) \tag{18}$$

so $\vec{s}_3 \in \text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\})$. Hence, we conclude that $\text{Span}(\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}) = \text{Span}(\{\vec{q}_1, \vec{q}_2, \vec{q}_3\})$.

2. Orthonormal Matrices and Projections

A matrix A has orthonormal columns, $\vec{a}_i$, if they are:

- Orthogonal (ie. $\langle \vec{a}_i, \vec{a}_j \rangle = \vec{a}_j^\top \vec{a}_i = 0$ when $i \neq j$)
- Normalized (ie. vectors with length equal to 1, $\|\vec{a}_i\| = 1$). This implies that $\|\vec{a}_i\|_2 = \langle \vec{a}_i, \vec{a}_i \rangle = \vec{a}_i^\top \vec{a}_i = 1$.

Let's consider the following wide matrix $A \in \mathbb{R}^{3\times 2}$:

$$A = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \tag{19}$$

(a) **Show that the matrix A has orthonormal columns.**

Solution: We must show that the columns of A satisfy both of properties listed above. Checking the norm of our columns:

$$\|\vec{a}_1\| = \left\| \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \right\| = \sqrt{\frac{1}{3}^2 + \frac{2}{3}^2 + \frac{2}{3}^2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{1} = 1 \tag{20}$$

$$\|\vec{a}_2\| = \left\| \begin{bmatrix} \frac{-2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix} \right\| = \sqrt{\frac{-2}{3}^2 + \frac{-1}{3}^2 + \frac{2}{3}^2} = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{1} = 1 \tag{21}$$

Now, let's check that our columns are orthogonal:

$$\langle \vec{a}_1, \vec{a}_2 \rangle = \vec{a}_2^\top \vec{a}_1 \tag{22}$$

$$= \frac{1}{3}\begin{bmatrix} -2 & -1 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \tag{23}$$

$$= \frac{1}{9}(-2 + -2 + 4) \tag{24}$$

$$= 0 \tag{25}$$

Thus our matrix A has orthogonal columns.

(b) **Now, calculate $A^\top A$.**

Solution:

$$A^\top A = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}^\top \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \tag{26}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \tag{27}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \tag{28}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{29}$$

$$= I_{2\times 2} \tag{30}$$

(c) **Calculate $AA^\top$ and compare your results with the previous part.**

Solution:

$$AA^\top = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}^\top \tag{31}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix} \tag{32}$$

$$= \frac{1}{9}\begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix} \tag{33}$$

Notice that this matrix is not full rank (since A only had a column rank of 2) and the third column can be seen as the second column minus the first column scaled by 2. This shows that that if a tall matrix has orthonormal columns then $A^\top A = I_{m\times m}$ where m is the number of columns but $AA^\top \neq I_{n\times n}$ where n is the number of rows of the tall matrix.

(d) Again, suppose we are working with same A matrix. Suppose that we wanted to project $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ onto the subspace spanned by the columns of A. **Show that the projection can be simplified to $AA^\top\vec{y}$ and then calculate the actual projection.**

Solution: Recall from 16A, that in order to project onto the column space of a matrix we use the least squares formula. By applying this result, we have that

$$\text{proj}_{\text{Col}(A)}(\vec{y}) = A\hat{\vec{x}} = A\left(A^\top A\right)^{-1} A^\top \vec{y} \tag{34}$$

Plugging in the result from part **2.**b,

$$\text{proj}_{\text{Col}(A)}(\vec{y}) = A\left(\underbrace{A^\top A}_{I_{m\times m}}\right)^{-1} A^\top \vec{y} \tag{35}$$

$$= AA^\top \vec{y} \tag{36}$$

If we calculate the projection we get:

$$\text{proj}_{\text{Col}(A)}(\vec{y}) = AA^\top \vec{y} \tag{37}$$

$$= \frac{1}{9} \begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \\ -9 \end{bmatrix} \quad (38)$$

$$= \begin{bmatrix} 7 \\ 2 \\ -10 \end{bmatrix} \quad (39)$$

(e) **(PRACTICE) Show if $A \in \mathbb{R}^{n \times n}$ is an orthonormal matrix then the columns, $\vec{a}_i$, form a basis for $\mathbb{R}^n$.**

Solution: Recall that, if we would like to show that a set of vectors are linearly independent, then the only β_i's satisfying

$$\beta_1 \vec{a}_1 + \beta_2 \vec{a}_2 + \ldots + \beta_n \vec{a}_n = \vec{0} \quad (40)$$

would be $\beta_i = 0$ for $i = 1$ to $i = n$. To show that $\beta_i = 0$ for the given instance, we can left multiply eq. (40) by $\vec{a}_i^\top$ (for any $i = 1$ to $i = n$):

$$\vec{a}_i^\top (\beta_1 \vec{a}_1 + \beta_2 \vec{a}_2 + \ldots + \beta_n \vec{a}_n) = \vec{a}_i^\top \vec{0} \quad (41)$$

$$\sum_{j=1}^{n} \beta_j \vec{a}_i^\top \vec{a}_j = 0 \quad (42)$$

$$\beta_i \underbrace{\vec{a}_i^\top \vec{a}_i}_{1} = 0 \quad (43)$$

$$\implies \beta_i = 0 \quad (44)$$

where we get to eq. (43) by using the fact that $\vec{a}_i^\top \vec{a}_j = 0$ for $i \neq j$. Hence, $\beta_i = 0$ for $i = 1$ to $i = n$.

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