1. Changing behavior through feedback

In this question, we discuss how feedback control can be used to change the effective behavior of a system.

(a) Consider the scalar system:

\[ x[i + 1] = 0.9x[i] + u[i] + w[i] \]  \hspace{1cm} (1)

where \( u[i] \) is the control input we get to apply based on the current state and \( w[i] \) is the external disturbance, each at time \( i \).

Is the system stable? If \( |w[i]| \leq \epsilon \), what can you say about \( |x[i]| \) at all times \( i \) if you further assume that \( u[i] = 0 \) and the initial condition \( x[0] = 0 \)? How big can \( |x[i]| \) get?

Solution: The system is stable, as \( \lambda = 0.9 \implies |\lambda| < 1 \). We can say that \( |x[i]| \) is bounded at all time if the disturbance is bounded. Unrolling the system’s recursion and extrapolating the general form,

\[ x[0] = 0 \]  \hspace{1cm} (2)
\[ x[1] = w[0] \]  \hspace{1cm} (3)
\[ x[2] = 0.9w[0] + w[1] \]  \hspace{1cm} (4)
\[ x[3] = 0.9^2w[0] + 0.9w[1] + w[2] \]  \hspace{1cm} (5)
\[ \vdots \]  \hspace{1cm} (6)
\[ x[i] = \sum_{k=0}^{i-1} 0.9^k w[i-k-1]. \]  \hspace{1cm} (7)

We can check that this form works by plugging it into our recursion:

\[ x[i + 1] = 0.9x[i] + w[i] = 0.9 \left( \sum_{k=0}^{i-1} 0.9^k w[i-k-1] \right) + w[i] \]  \hspace{1cm} (8)
\[ = \sum_{k=0}^{i-1} 0.9^{k+1} w[i-k-1] + w[i] = \sum_{k=0}^{i} 0.9^k w[i-k] \]  \hspace{1cm} (9)

which is exactly what our formula predicts. So,

\[ |x[i]| = \left| \sum_{k=0}^{i-1} 0.9^k w[i-k-1] \right| \leq \sum_{k=0}^{i-1} |0.9^k w[i-k-1]| \leq \sum_{k=0}^{i-1} 0.9^k \epsilon. \]  \hspace{1cm} (10)

In the limit as \( i \to \infty \), by the geometric series formula,

\[ |x[i]| \leq \frac{\epsilon}{1-0.9} = 10\epsilon \]  \hspace{1cm} (11)
(b) Suppose that we decide to choose a control law $u[i] = fx[i]$ to apply in feedback. Given a specific $\lambda$, you want the system to behave like:

$$x[i + 1] = \lambda x[i] + w[i]? \quad (12)$$

To do so, how would you pick $f$?

*NOTE:* In this case, $w[i]$ can be thought of like another input to the system, except we can’t control it.

**Solution:** We can control the system to have any value of $\lambda$, as long as we’re not limited on the values of $f$.

$$x[i + 1] = 0.9x[i] + fx[i] + w[i] = \lambda x[i] + w[i]. \quad (13)$$

Fitting terms, $f = \lambda - 0.9$. Note we can get a $\lambda > 1$ if we so desire; there is nothing stopping us from putting arbitrarily big/small $\lambda$ by the choice of $f$.

(c) For the previous part, which $f$ would you choose to minimize how big $|x[i]|$ can get?

**Solution:** From eq. (12), in order to have the minimum bound on $|x[i]|$, $\lambda = 0$. To get this $\lambda$, $f = -0.9$. In the limit as $i \to \infty$ in this case,

$$|x[i]| \leq \frac{\epsilon}{1 - 0} = \epsilon \quad (14)$$

The minimum bound on $|x(i)| = \epsilon$ is the same bound as on the disturbance.

(d) What if instead of a 0.9, we had a 3 in the original eq. (1). Would system stability change? Would our ability to control $\lambda$ change?

**Solution:** If our system were now,

$$x[i + 1] = 3x[i] + u[i] + w[i], \quad (15)$$

the system would no longer be stable. However, we can still choose any $\lambda$ using closed loop feedback. In this case, $f = \lambda - 3$. 

© UCB EECS 16B, Spring 2024. All Rights Reserved. This may not be publicly shared without explicit permission.
2. Eigenvalue Placement in Discrete Time

Recall that, for a discrete linear system to be stable, we require that all of the eigenvalues of $A$ in $\vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i]$ must have magnitude less than 1.

Consider the following linear discrete time system

$$\vec{x}[i + 1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vec{u}[i] + \vec{w}[i]$$  \hspace{1cm} (16)

(a) **Is the system given in eq. (16) stable?**

**Solution:** For notation’s sake, let’s write the system in the familiar form

$$\vec{x}[i + 1] = A\vec{x}[i] + \vec{b}\vec{u}[i] + \vec{w}[i]$$  \hspace{1cm} (17)

where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$  \hspace{1cm} (18)

We have to calculate the eigenvalues of matrix $A$. Doing so, we find:

$$\det (A - \lambda I) = 0 \implies \lambda_1 = 1, \lambda_2 = -2$$  \hspace{1cm} (19)

Since there exists a $\lambda$ such that $|\lambda| \geq 1$ (in fact, both $\lambda_1$ and $\lambda_2$ satisfy this inequality), the system is unstable.

(b) We can attempt to stabilize the system by implementing closed loop feedback. That is, we choose our input $\vec{u}[i]$ so that the system is stable. **If we were to use state feedback as in eq. (20), what is an equivalent representation for this system? Write your answer as $\vec{x}[i + 1] = A_{CL}\vec{x}[i]$ for some matrix $A_{CL}$.**

$$u[i] = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \vec{x}[i]$$  \hspace{1cm} (20)

**HINT:** If you’re having trouble parsing the expression for $u[i]$, note that $[f_1 \\ f_2]$ is a row vector, while $\vec{x}[i]$ is a column vector. What happens when we multiply a row vector with a column vector like this?

**Solution:** The closed loop system using state feedback has the form

$$\vec{x}[i + 1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vec{u}[i]$$  \hspace{1cm} (21)

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left( \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \right)$$  \hspace{1cm} (22)

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \vec{x}[i]$$  \hspace{1cm} (23)

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$$  \hspace{1cm} (24)

$$= \begin{bmatrix} f_1 & 1 + f_2 \\ 2 & -1 \end{bmatrix} \vec{x}[i].$$  \hspace{1cm} (25)
(c) Find the appropriate state feedback constants, \( f_1, f_2 \), that place the eigenvalues of the state space representation matrix at \( \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2} \).

Solution: From the previous part we have computed the closed loop system as
\[
\vec{x}[i + 1] = \begin{bmatrix} f_1 & 1 + f_2 \\ \frac{1}{2} & -1 \end{bmatrix} \vec{x}[i] \tag{26}
\]

Thus, finding the eigenvalues of the above system we have
\[
0 = \det (A - \lambda I) \tag{27}
\]
\[
= \det \left( \begin{bmatrix} f_1 - \lambda & 1 + f_2 \\ \frac{1}{2} & -1 - \lambda \end{bmatrix} \right) \tag{28}
\]
\[
= \lambda^2 + (1 - f_1)\lambda + (-f_1 - 2f_2 - 2) \tag{29}
\]

We want to place the eigenvalues at \( \lambda_1 = -\frac{1}{2} \) and \( \lambda_2 = \frac{1}{2} \). This means that we should choose the constants \( f_1 \) and \( f_2 \) so that the characteristic equation is
\[
0 = \left( \lambda - \frac{1}{2} \right) \left( \lambda + \frac{1}{2} \right) = \lambda^2 - \frac{1}{4} = \lambda^2 + 0\lambda - \frac{1}{4} \tag{30}
\]

Thus, we can match the coefficients of \( \lambda \) in the polynomial above, which indicates we should choose \( f_1 \) and \( f_2 \) satisfying the following system of equations:
\[
0 = 1 - f_1 \tag{31}
\]
\[
-\frac{1}{4} = -f_1 - 2f_2 - 2 \tag{32}
\]

We can solve this two variable, two equation system and find that \( f_1 = 1, f_2 = -\frac{11}{8} \).

Alternatively, we know what the eigenvalues are; we can plug in each \( \lambda \) into characteristic polynomial, and doing so will yield the same system of equations in \( f_1, f_2 \).

(d) Is the system now stable in closed-loop, using the control feedback coefficients \( f_1, f_2 \) that we derived above?

Solution: Yes, the closed loop system has eigenvalues \( \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2} \), which means that \( A_{CL} \) satisfies our condition that all of its eigenvalues have magnitude less than 1.

Contributors:
• Neelesh Ramachandran.
• Anant Sahai.
• Regina Eckert.
• Kumar Krishna Agrawal.
• Anish Muthali.
• Ioannis Konstantakopoulos.
• John Maidens.
• Druv Pai.