1. **Gram-Schmidt Algorithm**

Let’s apply Gram-Schmidt orthonormalization to a set of three linearly independent vectors \( \{ \vec{s}_1, \vec{s}_2, \vec{s}_3 \} \).

(a) Find unit vector \( \vec{q}_1 \) such that \( \text{span}(\{ \vec{q}_1 \}) = \text{span}(\{ \vec{s}_1 \}) \).

(b) Given \( \vec{q}_1 \) from the previous step, find \( \vec{q}_2 \) such that \( \text{span}(\{ \vec{q}_1, \vec{q}_2 \}) = \text{span}(\{ \vec{s}_1, \vec{s}_2 \}) \) and \( \vec{q}_2 \) is orthogonal to \( \vec{q}_1 \).

What would happen if \( \{ \vec{s}_1, \vec{s}_2, \vec{s}_3 \} \) were not linearly independent, but rather \( \vec{s}_1 \) were a multiple of \( \vec{s}_2 \)?

(c) Now given \( \vec{q}_1 \) and \( \vec{q}_2 \) in the previous steps, find \( \vec{q}_3 \) such that \( \text{span}(\{ \vec{q}_1, \vec{q}_2, \vec{q}_3 \}) = \text{span}(\{ \vec{s}_1, \vec{s}_2, \vec{s}_3 \}) \), and \( \vec{q}_3 \) is orthogonal to both \( \vec{q}_1 \) and \( \vec{q}_2 \), and finally \( \| \vec{q}_3 \| = 1 \).

(d) Let’s extend this algorithm to \( n \) linearly independent vectors. That is, given an input \( \{ \vec{s}_1, \ldots, \vec{s}_n \} \), write the algorithm to calculate the orthonormal set of vectors \( \{ \vec{q}_1, \ldots, \vec{q}_n \} \), where \( \text{span}(\{ \vec{s}_1, \ldots, \vec{s}_n \}) = \text{span}(\{ \vec{q}_1, \ldots, \vec{q}_n \}) \).

*Hint:* How would you calculate the \( i \)th vector, \( \vec{q}_i \)?
2. The Order of Gram-Schmidt

If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the set of vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Perform Gram-Schmidt on these vectors first in the order $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

(b) Now perform Gram-Schmidt on these vectors in the order $\vec{v}_3$, $\vec{v}_2$, $\vec{v}_1$. Do you get the same result?

Contributors:

• Regina Eckert.
• Druv Pai.