1. System Identification by Means of Least Squares

(a) Consider the scalar discrete-time system

\[ x[i + 1] = ax[i] + bu[i] + w[i] \]  \hspace{1cm} (1)

Where the scalar state at timestep \( i \) is \( x[i] \), the input applied at timestep \( i \) is \( u[i] \) and \( w[i] \) represents some (small) external disturbance that also participated at timestep \( i \) (which we cannot predict or control, it’s a purely random disturbance).

Assume that you have measurements for the states \( x[i] \) from \( i = 0 \) to \( \ell \) and also measurements for the controls \( u[i] \) from \( i = 0 \) to \( \ell - 1 \). Further assume \( \ell \geq 2 \).

Show that we can set up a linear system as in eq. (2) to find constants \( a \) and \( b \). How do we solve this system?

\[
\begin{bmatrix}
  x[1] \\
  x[2] \\
  \vdots \\
  x[\ell]
\end{bmatrix}
\approx
\begin{bmatrix}
  x[0] & u[0] \\
  \vdots & \vdots \\
  x[\ell-1] & u[\ell-1]
\end{bmatrix}
\begin{bmatrix}
  a \\ b \\
\end{bmatrix}
\]  \hspace{1cm} (2)
(b) What if there were now two distinct scalar inputs to a scalar system

\[ x[i + 1] = ax[i] + b_1u_1[i] + b_2u_2[i] + w[i] \]  

(3)

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters \( a, b_1, b_2 \).

(c) What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?

(d) Now consider the two dimensional state case with a single input.

\[ \vec{x}[i + 1] = \begin{bmatrix} x_1[i + 1] \\ x_2[i + 1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \]  

(4)

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters \( a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 \)? Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts). Hint: What work/computation can we reuse across the two problems?
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