Discussion 8B

1. Introduction to Discrete Time Difference Equations

In this question, we discuss how to think about systems represented in discrete time, as well as the similarities and differences with systems in continuous time.

(a) Consider the scalar system:

$$x[i+1] = 3x[i] + 2u[i] \tag{1}$$

where $x[i]$ is the state parameter of the system and $u[i]$ is the input to the system, each at time i. In this question, we will solve this equation to obtain the closed form solution, ie. an expression for $x[i]$ that only depends on i and $u[i]$. Suppose that the initial condition of this system is $x[0] = 1$.

i. The solution method is very similar to first order ODEs. **First, solve for the homogeneous solution of the difference equation. What kind of solution should $x[i]$ be?** *(HINT: You should guess something similar to $e^{\lambda t}$ which is what we guessed for ODEs.)*

Solution: Solving for the homogeneous case means that we ignore the inputs that do not depend on $x[i]$. Therefore, our original equation becomes:

$$x[i+1] = 3x[i] \tag{2}$$

Further, we guess that the solution should be $k\lambda^i$. This is not immediately obvious, but the idea is similar to that of the continuous time case. We want $x[i]$ to be a function that has the same term on the left and right hand sides of the equation, similar to how $\frac{dx(t)}{dt} = \lambda x(t)$ gives the intuition/reason for choosing $x(t) = e^{\lambda t}$: $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$. Therefore, we get that:

$$k\lambda^{i+1} = 3 * k\lambda^i \tag{3}$$

$$\lambda = 3 \tag{4}$$

Therefore, $x_h[i] = k * 3^i$. This is a simple enough fact, but you will study the meaning and importance of the constant λ when we study system stability which is related to the question asked in part a)iv). (In this case $\lambda = 3$). Note that the constant k can be zero. If that is the case, then there is no homogeneous solution (and the equation above simply becomes $0 = 0$). Another way to approach the solution is by using the recurrence relation defined by the system equation 2. Namely, we know that $x[1] = 3x[0]$, $x[2] = 3x[1] = 9x[0]$, and so on, which we can use to conclude that $x_h[i] = x[0]3^i$. This also shows that the constant k is in fact the initial condition, just like in the continuous time case.

ii. Now that we have the homogeneous solution, **solve for the particular solution by setting $x[0] = 0$.** *(HINT: Start by solving for $x[1]$, then $x[2]$ using your value of $x[1]$, and so on. Do you see a pattern emerging?)*

Solution: Following the hint:

$$x[1] = 3x[0] + 2u[0] = 2u[0] \tag{5}$$

$$x[2] = 3x[1] + 2u[1] = 3(2u[0]) + 2u[1] = 6u[0] + 2u[1] \tag{6}$$

$$x[3] = 3x[2] + 2u[2] = 18u[0] + 6u[1] + 2u[2] \tag{7}$$

$$\ldots \tag{8}$$

The pattern is a little tricky to see, but notice that for each $x[i]$, we have a summation of the inputs from time 0 to time $i-1$ and that each one is a scaled version of the other. Namely, the common factor between all the $u[i]$ is a multiple of 3 which increases as the time decreases. Therefore,

$$x_p[i] = \sum_{n=0}^{i-1} 3^{i-1-n} * 2u[n] \tag{9}$$

is the particular solution.

iii. Using your answers for the previous parts and the initial condition given in part (a), **find the solution for** $x[i]$.

Solution:

$$x[i] = x_h[i] + x_p[i] = k3^i + \sum_{n=0}^{i-1} 3^{i-1-n} * 2u[n] \tag{10}$$

Using the initial condition, we get that

$$x[0] = k + \sum_{n=0}^{-1} 3^{i-1-n} * 2u[n] = k = 1 \tag{11}$$

$$k = 1 \tag{12}$$

Thus, our full solution is

$$x[i] = 3^i + \sum_{n=0}^{i-1} 3^{i-1-n} * 2u[n] \tag{13}$$

Now that we have this general solution, we can compare it to the closed form solution of the 1st order ODE that was shown in Note 2 (using $t_0 = 0$). Namely,

$$x(t) = x(0)e^{\lambda t} + \int_0^t e^{\lambda(t-\theta)} * bu(\theta)d\theta \tag{14}$$

Of course, the solutions are not identical, but notice the similarities. The first term is the initial condition times the exponential in time and the second term is a summation on the input u.

Finally, note that if we replace 3 with an arbitrary constant a and 2 with an arbitrary constant b, we get the solution to the general difference equation,

$$x[i+1] = ax[i] + bu[i] \tag{15}$$

with some initial condition $x[0]$,

$$x[i] = a^i x[0] + \sum_{n=0}^{i-1} a^{i-1-n} * bu[n] \tag{16}$$

iv. Now, suppose that $u[i] = b$ for all time. **What happens to $x[i]$ as i goes to infinity?**

Solution:

$$x[i] = 3^i + \sum_{n=0}^{i-1} 3^{i-1-n} * 2b \tag{17}$$

$$= 3^i + 2b \sum_{n=0}^{i-1} 3^{i-1-n} \tag{18}$$

$$= 3^i + 2b \sum_{n=0}^{i-1} 3^n \tag{19}$$

When taking the limit as $n \to \infty$, both 3^i and the summation will diverge which means that $x[i]$ also diverges. Note that the summation is a geometric series with constant 3, which makes the series diverge. So,

$$\lim_{i \to \infty} x[i] = \infty \tag{20}$$

What if 3 were instead some other number? What inequality would that number have to satisfy so that $\lim_{i\to\infty} x[i]$ converges given that $u[i] = b$ for all time? If $u[i]$ were instead an arbitrary but bounded function, would that range of numbers still ensure that $x[i]$ converges?

(b) In this part, we will solve for the impulse and step response of the scalar first order difference equation. Suppose the system equation is:

$$x[i+1] = ax[i] + bu[i] \tag{21}$$

Suppose that the state system's initial condition is $x[0] = 0$; that is, suppose the system starts at rest.

i. **Solve for the impulse response of the system, $h[i]$, by setting $u[i] = \delta[i]$.** The impulse is defined as

$$\delta[i] = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases}$$

Solution: The system equation becomes

$$x[i+1] = ax[i] + b\delta[i] \tag{22}$$

Now, since we are in discrete time, we can solve for each value of i and see if a pattern emerges:

$$x[1] = ax[0] + b\delta[0] = a * 0 + b * 1 = b \tag{23}$$

$$x[2] = ax[1] + b\delta[1] = a * b + b * 0 = ab \tag{24}$$

$$x[3] = ax[2] + b\delta[2] = a * (ab) + b * 0 = a^2 b \tag{25}$$

$$\ldots \tag{26}$$

The pattern is clear. After we pass time $i = 1$, the impulse will never contribute to the system again such that the system will only depend on the initial "momentum" that it gained from the impulse. Thus, we get that the impulse response is $h[i] = a^{i-1}b$.

ii. **Solve for the step response of the system, $f[i]$, by setting $u[i] = \text{step}[i]$.** The step function is defined as

$$\text{step}[i] = \begin{cases} 1 & i \geq 0 \\ 0 & i < 0 \end{cases}$$

Solution: The system equation becomes

$$x[i+1] = ax[i] + b * \text{step}[i] \tag{27}$$

Now, since we are in discrete time, we can solve for each value of i and see if a pattern emerges:

$$x[1] = ax[0] + b * \text{step}[0] = a * 0 + b * 1 = b \tag{28}$$
$$x[2] = ax[1] + b * \text{step}[1] = a * b + b * 1 = ab + b \tag{29}$$
$$x[3] = ax[2] + b * \text{step}[2] = a * (ab + b) + b * 1 = a^2b + ab + b \tag{30}$$
$$x[4] = ax[3] + b * \text{step}[3] = a^3b + a^2b + ab + b \tag{31}$$
$$\ldots \tag{32}$$

The pattern is clear. After we pass time $i = 1$, the step function will continue to contribute to the system such that the system will depend on the state parameter and the continuous input. Thus, putting the above expressions into a summation, we get that the impulse response is

$$f[i] = \sum_{n=1}^{i} a^{n-1}b \tag{33}$$

Determining the system response to a certain input is an important topic in signals and systems, which you will study if you take EE 120.

(c) Now suppose that we have a second order difference equation.

$$x_1[i] - ax_1[i-1] - bx_1[i-2] = cu[i] \tag{34}$$

We could solve this equation using the same method that we used in part (a), but we'll convert it into a vector difference equation since we already know how to solve first order difference equations.

Convert the above equation into a system of equations by setting $x_2[i] = x_1[i-1]$ *(HINT: We want the state variables, x_1 and x_2, on the left hand side of the vector difference equation to depend on time i and the right hand side to depend on time $i-1$.)*

Solution: By doing the above, we can use the fact that $x_2[i-1] = x_1[i-2]$ to convert the original equation into equation (35) and we also have the second equation given by the substitution. So, we have two equations:

$$x_1[i] = ax_1[i-1] + bx_2[i-1] + cu[i] \tag{35}$$
$$x_2[i] = x_1[i-1] \tag{36}$$

We substitute x_2 into the system equation in this way because we want to get a system of first order difference equations. So, the left hand side state variables depend on time i and the right hand side on time $i-1$. Therefore, we get that:

$$\vec{x}[i] = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \vec{x}[i-1] + \begin{bmatrix} cu[i] \\ 0 \end{bmatrix} \tag{37}$$

where $\vec{x}[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}$. Solving systems of difference equations is somewhat similar to solving a system of differential equations. It's easier in the sense that the solution does not involve integration and the homogeneous part of the solution does not need to be diagonal like $e^{\Lambda t}$.

For completion, suppose that the system of difference equations is of the form:

$$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i] \tag{38}$$

Then, the solution is:

$$\vec{x}[i] = A^i x[0] + \sum_{n=0}^{i-1} A^{i-1-n} B\vec{u}[n] \tag{39}$$

Compare this to the answer from part a)iii). We can ask the same question posed in part a)iv); what properties must the A matrix satisfy for $\vec{x}[i]$ to converge as $i \to \infty$?

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