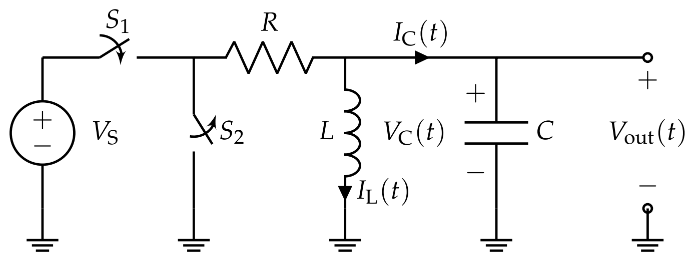
Discussion 8A

1. RLC Circuit with Vector Differential Equations

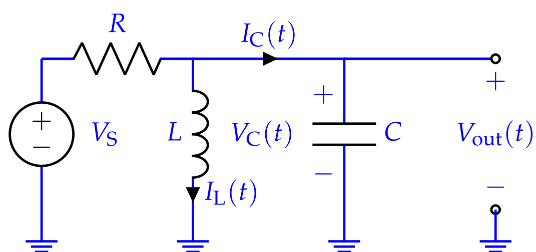
Consider the following circuit fed by a constant voltage source V_S.



The switch S_1, open for $t < 0$, closes at $t = 0$, and the switch S_2, closed for $t < 0$, opens at $t = 0$. Assume $V_C(0) = 0$ and $I_L(0) = 0$.

(a) **Derive a set of two differential equations, one for $I_L(t)$, the current through the inductor, and one for $V_C(t)$, the voltage across the capacitor.** Write your answer in terms of R, L, C, V_S, and constants.

Solution: The circuit appears as follows:



From Ohm's law for the resistor and KCL at the node with voltage V_C, we have:

$$\frac{V_S - V_C}{R} = I_L + I_C \tag{1}$$

Substituting in the I-V relationship for capacitors in the previous equation, we now have:

$$\frac{V_S - V_C}{R} = I_L + C\frac{\mathrm{d}V_C}{\mathrm{d}t} \tag{2}$$

$$\frac{V_S}{RC} - \frac{V_C}{RC} = \frac{I_L}{C} + \frac{\mathrm{d}V_C}{\mathrm{d}t} \tag{3}$$

$$\frac{V_S}{RC} - \frac{V_C}{RC} - \frac{I_L}{C} = \frac{\mathrm{d}V_C}{\mathrm{d}t} \tag{4}$$

Now, we can notice that $V_C = V_L$ as the inductor and capacitor are in parallel. From the inductor I-V relationship, we have:

$$L\frac{\mathrm{d}I_L}{\mathrm{d}t} = V_L = V_C \tag{5}$$

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{V_C}{L} \tag{6}$$

In summary, the two differential equations are as follows.

$$\frac{\mathrm{d}V_C}{\mathrm{d}t} = -\frac{V_C}{RC} - \frac{I_L}{C} + \frac{V_S}{RC} \tag{7}$$

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{V_C}{L} \tag{8}$$

(b) **Using your answers from the previous part, create a vector differential equation with the state vector being $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$.** Write your answers in terms of R, L, C, V_S, and constants.

Solution: The previous part has the following differential equations.

$$\frac{\mathrm{d}V_C}{\mathrm{d}t} = -\frac{V_C}{RC} - \frac{I_C}{C} + \frac{V_S}{RC} \tag{9}$$

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{V_C}{L} \tag{10}$$

Stacking the above equations into matrix-vector form, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} V_S \tag{11}$$

(c) Regardless of your answer to the previous part, suppose the vector differential equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{\vec{b}} V_S \tag{12}$$

First, find the eigenvalues of the matrix A.

Solution: To find our eigenvalues, we use the method used in EECS 16A:

$$A\vec{v} = \lambda\vec{v} \tag{13}$$

$$A\vec{v} - \lambda\vec{v} = 0 \tag{14}$$

$$(A - \lambda I_2)\vec{v} = 0 \tag{15}$$

$$\det(A - \lambda I_2) = 0 \tag{16}$$

$$\det\left(\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 \tag{17}$$

$$\det\left(\begin{bmatrix} -4-\lambda & -6 \\ \frac{1}{2} & -\lambda \end{bmatrix}\right) = 0 \tag{18}$$

$$\lambda^2 + 4\lambda + 3 = 0 \tag{19}$$

Therefore our eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$.

(d) **Next, find the eigenvectors that will form your V basis.**

Solution: Recall, that the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = -1$. We find the corresponding eigenvectors by plugging in our eigenvalues:

For $\lambda_1 = -3$:

$$\left(\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right)\vec{v}_1 = 0 \tag{20}$$

$$\left(\begin{bmatrix} -1 & -6 \\ \frac{1}{2} & 3 \end{bmatrix}\right)\vec{v}_1 = 0 \tag{21}$$

From inspection or Gaussian Elimination, we find that $\vec{v}_1 = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.

For $\lambda_1 = -1$:

$$\left(\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\vec{v}_2 = 0 \tag{22}$$

$$\left(\begin{bmatrix} -3 & -6 \\ \frac{1}{2} & 1 \end{bmatrix}\right)\vec{v}_2 = 0 \tag{23}$$

$$\tag{24}$$

From inspection or Gaussian Elimination, we find that $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(e) **Now, in order to diagnolize the system, write A in terms of V, V^{-1}, and Λ.** *(HINT: For a 2×2 real matrix, the inverse of that matrix is $V^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.)*

Solution: From Note 4, we know that:

$$\underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -6 & -2 \\ 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}}_{V^{-1}} \tag{25}$$

(f) **With $\vec{x}(0) = \vec{0}$, solve for $\vec{x}(t)$ and find the asymptotic/steady-state behavior as $t \to \infty$.** *(HINT: Use the information from the previous part to perform a change of basis that simplifies the state equations.)*

Solution: We can define $\vec{z}(t)$ to be the representation of $\vec{x}(t)$ in the eigenbasis which, in this case, is the V-basis. In other words, $\vec{z}(t) = V^{-1}\vec{x}(t)$ and $\vec{x}(t) = V\vec{z}(t)$. The matrix V takes a vector from the eigenbasis and converts it into the standard basis. Applying this, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}V_S \tag{26}$$

$$V^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t)\right) = \underbrace{V^{-1}AV}_{\Lambda}\vec{z}(t) + V^{-1}\vec{b}V_S \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \Lambda\vec{z}(t) + \underbrace{\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{V^{-1}\vec{b}} V_S \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} V_S \tag{29}$$

The initial condition is $\vec{z}(0) = V^{-1}\vec{x}(0) = V^{-1}\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We can solve these equations one row at a time, either by using substitution or the general first order differential equation solution. We use the latter approach. Also, note that since both the initial conditions are equal to 0, there is no homogeneous term, namely, the homogeneous solution will be 0. Solve the first row equation for $z_1(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}z_1 = -3z_1 - V_S \tag{30}$$

$$\implies z_1(t) = -V_S\mathrm{e}^{-3t}\int_0^t \mathrm{e}^{3\theta}\,\mathrm{d}\theta \tag{31}$$

$$= -V_S\mathrm{e}^{-3t}\frac{\mathrm{e}^{3\theta}}{3}\Big|_{\theta=0}^{\theta=t} \tag{32}$$

$$= -V_S\mathrm{e}^{-3t}\frac{\mathrm{e}^{3t}-1}{3} \tag{33}$$

$$= V_S\frac{\mathrm{e}^{-3t}-1}{3} \tag{34}$$

An alternative way to solve this is by inspection. We know that our solution should have an exponential term and a constant term, seeing that the input voltage is constant. We also know that $z_1(0) = 0$ which means that our guess for the solution is $z_1(t) = B\mathrm{e}^{-at} - B$. Plug it into the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}z_1 = -3z_1 - V_S \tag{35}$$

$$\implies -aB\mathrm{e}^{-at} = -3B\mathrm{e}^{-at} + 3B - V_S \tag{36}$$

$$3B - V_S = 0 \implies B = \frac{V_S}{3} \tag{37}$$

$$-a\mathrm{e}^{-at} = -3\mathrm{e}^{-at} \implies a = 3 \tag{38}$$

which yields $z_1(t) = \frac{V_S}{3}\mathrm{e}^{-3t} - \frac{V_S}{3} = V_S\frac{\mathrm{e}^{-3t}-1}{3}$. And now solve the second row equation for $z_2(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}z_2 = -z_2 + V_S \tag{39}$$

$$\implies z_2(t) = V_S\mathrm{e}^{-t}\int_0^t \mathrm{e}^{\theta}\,\mathrm{d}\theta \tag{40}$$

$$= V_S\mathrm{e}^{-t}(\mathrm{e}^t - 1) \tag{41}$$

$$= V_S(1 - \mathrm{e}^{-t}) \tag{42}$$

We can also solve this by inspection. Our guess for the solution is $z_2(t) = B\mathrm{e}^{-at} - B$. Plug it into the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}z_2 = -z_2 + V_S \tag{43}$$

$$\implies -aB\mathrm{e}^{-at} = -B\mathrm{e}^{-at} + B + V_S \tag{44}$$

$$B + V_S = 0 \implies B = -V_S \tag{45}$$

$$-ae^{-at} = -e^{-at} \implies a = 1 \tag{46}$$

which yields $z_2(t) = -V_S e^{-t} + -V_S = V_S(1 - e^{-t})$. Therefore,

$$\vec{z}(t) = V_S \begin{bmatrix} \frac{e^{-3t}-1}{3} \\ 1 - e^{-t} \end{bmatrix} \tag{47}$$

Now, we want to convert our answer back into the standard basis by multiplying $V\vec{z}(t)$:

$$\vec{x}(t) = V\vec{z}(t) = V_S \begin{bmatrix} -6 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{e^{-3t}-1}{3} \\ 1 - e^{-t} \end{bmatrix} = V_S \begin{bmatrix} 2\left(e^{-t} - e^{-3t}\right) \\ \frac{e^{-3t}-3e^{-t}+2}{3} \end{bmatrix} \tag{48}$$

This is our final answer. Lastly, taking a limit as $t \to \infty$, we have

$$\lim_{t\to\infty} \vec{x}(t) = V_S \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} \tag{49}$$

In particular, we can say that $\lim_{t\to\infty} V_C(t) = 0$, and that $\lim_{t\to\infty} I_L(t) = \frac{2}{3}V_S$.

We can verify that this result matches the expected circuit behavior in the steady state. In steady state, the capacitor behaves as an open circuit and the inductor as a short circuit. Therefore, the voltage drop across the capacitor and inductor will be zero since the current is shorted to ground. Thus, $\lim_{t\to\infty} V_C(t) = 0$. The current through the inductor will be the current of the resistor which is $\lim_{t\to\infty} I_L(t) = \frac{V_S}{R}$. From the values given in equation 12, $\frac{1}{C} = 6$ and $\frac{1}{RC} = 4$ which means that $\frac{1}{R} = \frac{2}{3}$. Thus, $\lim_{t\to\infty} I_L(t) = \frac{2}{3}V_S$.

