Discussion 7B

The following notes are useful for this discussion: Note 10, Note 11

1. System Identification by Means of Least Squares

(a) Consider the scalar discrete-time system

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{1}$$

Where the scalar state at timestep i is $x[i]$, the input applied at timestep i is $u[i]$ and $w[i]$ represents some (small) external disturbance that also participated at timestep i (which we cannot predict or control, it's a purely random disturbance).

Assume that you have measurements for the states $x[i]$ from $i = 0$ to ℓ and also measurements for the controls $u[i]$ from $i = 0$ to $\ell - 1$. Further assume $\ell \geq 2$.

Show that we can set up a linear system as in eq. (2) to find constants a and b. How do we solve this system?

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \underbrace{\begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} \tag{2}$$

Solution: Our model is of the form

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{3}$$

where $w[i]$ is our error term and we are interested in a and b. Since we cannot predict the disturbance $w[i]$ (and therefore cannot have a parameter in our solution associated with the effect of the disturbance on our system), we will solve the adjusted equation in eq. (4).

$$x[i+1] \approx ax[i] + bu[i] \tag{4}$$

We have measurements from $i = 1$ to $i = m$, and so our least squares formulation is:

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \underbrace{\begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} \tag{5}$$

D is not necessarily a square matrix (it is tall), so we cannot invert it and solve for $\vec{p}$. Hence, we use least squares like previously mentioned. Thus, our best approximation for $\vec{p}$ is

$$\widehat{\vec{p}} = \left(D^\top D\right)^{-1} D^\top \vec{s} \tag{6}$$

Since we are using least squares, we can also group our estimation error (remember, $\widehat{\vec{p}} \neq \vec{p}$ necessarily) into $w[i]$.

(b) What if there were now two distinct scalar inputs to a scalar system

$$x[i+1] = ax[i] + b_1 u_1[i] + b_2 u_2[i] + w[i] \tag{7}$$

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a, b_1, b_2.

Solution: Our new model is of the form

$$x[i+1] = ax[i] + b_1 u_1[i] + b_2 u_2[i] + w[i] \tag{8}$$

where $w[i]$ is our error term and we are interested in a, b_1, b_2. As we did before, we will modify the system and drop the disturbance term, converting the equality to an approximation.

$$x[i+1] \approx ax[i] + b_1 u_1[i] + b_2 u_2[i] \tag{9}$$

As before, we have $[1, m]$ measurements, and so our least squares formulation is:

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \underbrace{\begin{bmatrix} x[0] & u_1[0] & u_2[0] \\ x[1] & u_1[1] & u_2[1] \\ \vdots & \vdots & \vdots \\ x[\ell-1] & u_1[\ell-1] & u_2[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix}}_{\vec{p}} \tag{10}$$

(c) **What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?**

Solution: We can take a look at the least squares formula, and think about what the possible failure points are.

$$\widehat{\vec{p}} = \left(D^\top D\right)^{-1} D^\top \vec{s}. \tag{11}$$

In this equation, the likely point of failure is the inversion of $D^\top D$; the other operations (matrix-matrix multiplications, matrix-vector multiplications) do not have the same issue.

$D^\top D$ might not be invertible when D has columns that are not linearly independent. For example, it could be because the inputs $\vec{u}_1$ and $\vec{u}_2$ are too similar, as if $\vec{u}_1 = \alpha \vec{u}_2$. We need these two inputs to be different and sufficiently varied so that least-squares does not fail.

(d) Now consider the two dimensional state case with a single input.

$$\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \tag{12}$$

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$? Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts). *Hint: What work/computation can we reuse across the two problems?*

Solution: We can rewrite eq. (12) as

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11}x_1[i] + a_{12}x_2[i] + b_1u[i] \\ a_{21}x_1[i] + a_{22}x_2[i] + b_2u[i] \end{bmatrix} \tag{13}$$

We can set up a problem to solve for a_{11}, a_{12}, b_1 (call this subsystem 1) and another problem to solve for a_{21}, a_{22}, b_2 (call this subsystem 2). We can rewrite the first row of eq. (13) as

$$x_1[i+1] = \begin{bmatrix} x_1[i] & x_2[i] & u[i] \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \end{bmatrix} \tag{14}$$

and likewise for the second row

$$x_2[i+1] = \begin{bmatrix} x_1[i] & x_2[i] & u[i] \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ b_2 \end{bmatrix} \tag{15}$$

To find the unknowns in subsystem 1, we can set up the following least squares problem:

$$\underbrace{\begin{bmatrix} x_1[1] \\ x_1[2] \\ \vdots \\ x_1[\ell] \end{bmatrix}}_{\vec{s}_1} \approx \underbrace{\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1 \\ \vdots & \vdots & \vdots \\ x_1[\ell-1] & x_2[\ell-1] & u[\ell-1] \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \end{bmatrix}}_{\vec{p}_1} \tag{16}$$

Now, to find the unknowns in subsystem 2, we can set up the following least squares problem:

$$\underbrace{\begin{bmatrix} x_2[1] \\ x_2[2] \\ \vdots \\ x_2[\ell] \end{bmatrix}}_{\vec{s}_2} \approx \underbrace{\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1 \\ \vdots & \vdots & \vdots \\ x_1[\ell-1] & x_2[\ell-1] & u[\ell-1] \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} a_{21} \\ a_{22} \\ b_2 \end{bmatrix}}_{\vec{p}_2} \tag{17}$$

Notice that $D_1 = D_2$. Hence, we can write $D = D_1 = D_2$, and we only need to compute $\left(D^\top D\right)^{-1} D^\top$ once. Hence, the solution for the ith subsystem (for $i \in \{1, 2\}$) is

$$\widehat{\vec{p}}_i = \left(D^\top D\right)^{-1} D^\top \vec{s}_i \tag{18}$$

Furthermore, we can horizontally stack the two separate problems for each subsystem as follows:

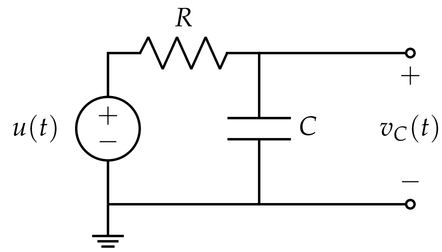
$$\underbrace{\begin{bmatrix} x_1[1] & x_2[1] \\ x_1[2] & x_2[2] \\ \vdots & \vdots \\ x_1[\ell] & x_2[\ell] \end{bmatrix}}_{S} \approx \underbrace{\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ \vdots & \vdots & \vdots \\ x_1[\ell-1] & x_2[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2 \end{bmatrix}}_{P} \tag{19}$$

Finally, solving this as a single least squares problem gives us

$$\widehat{P} = \left(D^\top D\right)^{-1} D^\top S \tag{20}$$

2. Stability Examples and Counterexamples

(a) Consider the circuit below with $R = 1\,\Omega$, $C = 0.5\,\text{F}$, and $u(t)$ is some function bounded between $-K$ and K for some constant $K \in \mathbb{R}$ (for example $K\cos(t)$). Furthermore assume that $v_C(0) = 0\,\text{V}$ (that the capacitor is initially discharged).



This circuit can be modeled by the differential equation

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = -2v_C(t) + 2u(t) \tag{21}$$

Show that the differential equation is always stable (that is, as long as the input $u(t)$ is bounded, $v_C(t)$ also stays bounded). Consider what this means in the physical circuit. *HINT: You may want to use the triangle inequality, i.e. $|a + b| \leq |a| + |b|$, and the triangle inequality for integrals, i.e. $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$. When we use $|\cdot|$ notation here, we will take this to mean the magnitude, rather than the absolute value (since we can be dealing with complex numbers).*

Solution: We can apply the integral solution for a nonhomogeneous differential equation to demonstrate boundedness of the solution. The general solution to $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t)$ is $x(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda(t-\theta)} bu(\theta)\,\mathrm{d}\theta$. Here, we can say that:

$$v_C(t) = v_C(0)e^{-2t} + \int_0^t e^{-2(t-\theta)} 2u(\theta)\,\mathrm{d}\theta \tag{22}$$

$$= v_C(0)e^{-2t} + 2\int_0^t e^{-2(t-\theta)} u(\theta)\,\mathrm{d}\theta \tag{23}$$

We wish to show $|v_C(t)| \leq M$ for all $t \geq 0$, where $M \in \mathbb{R}$ is some constant (this is another way to say that something is "bounded"). We can take the absolute value around eq. (23) as follows:

$$|v_C(t)| = \left|v_C(0)e^{-2t} + 2\int_0^t e^{-2(t-\theta)} u(\theta)\,\mathrm{d}\theta\right| \tag{24}$$

$$\leq \left|v_C(0)e^{-2t}\right| + \left|2\int_0^t e^{-2(t-\theta)} u(\theta)\,\mathrm{d}\theta\right| \tag{25}$$

$$\leq \left|v_C(0)e^{-2t}\right| + 2\int_0^t \left|e^{-2(t-\theta)} u(\theta)\right|\,\mathrm{d}\theta \tag{26}$$

$$= |v_C(0)|e^{-2t} + 2\int_0^t e^{-2(t-\theta)} |u(\theta)|\,\mathrm{d}\theta \tag{27}$$

where we use the traditional triangle inequality to obtain eq. (25) and the integral triangle inequality to obtain eq. (26). We know $v_C(0) = 0$, so the first term is 0. Even if it is nonzero, we may assume that it is some finite constant. Furthermore, $0 \leq e^{-2t} \leq 1$ for $t \geq 0$ (it is a decaying exponential). Hence, the $|v_C(0)|e^{-2t}$ term is bounded. Next, we are allowed to assume that

$|u(t)| \le K$ from the statement of the problem. This will let us obtain

$$|v_C(t)| \le 2\int_0^t \mathrm{e}^{-2(t-\theta)} \underbrace{|u(\theta)|}_{\le K} \,\mathrm{d}\theta \tag{28}$$

$$\le 2K\int_0^t \mathrm{e}^{-2(t-\theta)} \,\mathrm{d}\theta \tag{29}$$

$$= K\left(1-\mathrm{e}^{-2t}\right) \tag{30}$$

Because $\mathrm{e}^{-2t} \ge 0$, $1-\mathrm{e}^{-2t} \le 1$. Hence, $|v_C(t)| \le K$ so $v_C(t)$ is bounded.

(b) **(PRACTICE)** Now, suppose that in the circuit of part **2.a** we replaced the resistor with an inductor as in fig. 1.

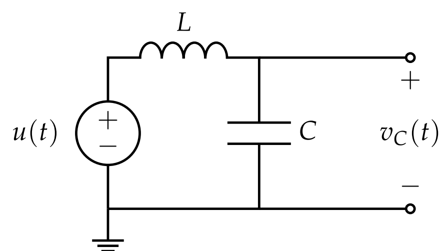


Figure 1: The original circuit with an inductor in place of the resistor.

Let $L = 1\,\mathrm{mH}$. **Repeat part 2.a for the new circuit (with an inductor). Consider the following process to arrive at the result:**

i. **Derive the system of differential equations using KCL, KVL, and NVA. Show that the system is** $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u(t)$ **with the initial condition being** $\begin{bmatrix} v_C(0) \\ i_L(0) \end{bmatrix} = \vec{0}$**.**

ii. **Solve the matrix differential equation, using diagonalization if needed. Show that the diagonalized system has a solution**

$$\vec{y}(t) = \begin{bmatrix} \frac{1}{2LC}\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}\int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta} u(\theta)\,\mathrm{d}\theta \\ \frac{1}{2LC}\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}\int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta} u(\theta)\,\mathrm{d}\theta \end{bmatrix} \tag{31}$$

where $\vec{y}(t) = V^{-1}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ **for change of basis matrix** V**.** You may use the fact that the eigenvalue, eigenvector pairs of $\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$ are $\left(\mathrm{j}\frac{1}{\sqrt{LC}}, \begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}\right)$ and $\left(-\mathrm{j}\frac{1}{\sqrt{LC}}, \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix}\right)$.

iii. **Apply a similar process from part 2.a to show that, if we have a bounded input** $u(t)$**, then the system can grow unboundedly.** When showing that a system is unstable, it suffices to choose a bounded $u(t)$ that makes the system unbounded. We can choose $u(t) = 2\cos\left(\frac{1}{\sqrt{LC}}\right) = \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t} + \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}$ [1]. *HINT: You may use the fact that* $i_L(t) = y_1(t) + y_2(t)$.

[1] The natural frequency of this system is $\omega_n = \frac{1}{\sqrt{LC}}$. If we excite this system at a period equal to the natural frequency, we can make it grow unboundedly. This is similar to pushing a swing at the same rate it swings, which makes it swing farther.

Hint: You might find it useful to revisit the process of generating the state-space equations for $v_C(t)$ and $i_L(t)$ as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.

Solution: 2.(b)i:

First, we begin forming the vector state-space equation, which involves relating $v_C(t)$ and $i_L(t)$ to their derivatives and the input voltage.

$$C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = i_C(t) = i_L(t) \tag{32}$$

$$\implies \frac{\mathrm{d}v_C(t)}{\mathrm{d}t} = \frac{1}{C}i_L(t) \tag{33}$$

$$L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = v_L(t) = u(t) - v_C(t) \tag{34}$$

$$\implies \frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = \frac{1}{L}v_L(t) = -\frac{1}{L}v_C(t) + \frac{1}{L}u(t) \tag{35}$$

Combining this info, we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{\vec{b}} u(t) \tag{36}$$

2.(b)ii:

This is not a diagonal system, so we have to diagonalize it first. We start by solving for the eigenvalues and eigenvectors of A:

$$\lambda_1 = \mathrm{j}\frac{1}{\sqrt{LC}} \qquad \vec{v}_1 = \begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \tag{37}$$

$$\lambda_2 = -\mathrm{j}\frac{1}{\sqrt{LC}} \qquad \vec{v}_1 = \begin{bmatrix} \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 \end{bmatrix} \tag{38}$$

Note that these eigenvalues are purely imaginary. This will be helpful later. Our change of basis matrix is $V = \begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} & \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}$, so we can define our change of basis as $\vec{y}(t) = V^{-1}\vec{x}(t)$. Note that the new diagonal system will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}(t) = \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix}\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{39}$$

$$= \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix}\vec{y}(t) + \left(\begin{bmatrix} -\mathrm{j}\sqrt{\frac{L}{C}} & \mathrm{j}\sqrt{\frac{L}{C}} \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}\right)u(t) \tag{40}$$

$$= \begin{bmatrix} \mathrm{j}\frac{1}{\sqrt{LC}} & 0 \\ 0 & -\mathrm{j}\frac{1}{\sqrt{LC}} \end{bmatrix}\vec{y}(t) + \begin{bmatrix} \frac{1}{2LC} \\ \frac{1}{2LC} \end{bmatrix}u(t) \tag{41}$$

so our system of equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}y_1(t) = \mathrm{j}\frac{1}{\sqrt{LC}}y_1(t) + \frac{1}{2LC}u(t) \tag{42}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}y_2(t) = -\mathrm{j}\frac{1}{\sqrt{LC}}y_2(t) + \frac{1}{2LC}u(t) \tag{43}$$

$$\tag{44}$$

Recall that $\vec{x}(0) = \vec{0}$, so $\vec{y}(t) = \vec{0}$ (where $\vec{0}$ is a vector of all zeros). Solving this differential equation now, we get

$$y_1(t) = \underbrace{y_1(0)}_{0}\,\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t} + \int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}(t-\theta)}\left(\frac{1}{2LC}u(\theta)\right)\mathrm{d}\theta \tag{45}$$

$$y_2(t) = \underbrace{y_2(0)}_{0}\,\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t} + \int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}(t-\theta)}\left(\frac{1}{2LC}u(\theta)\right)\mathrm{d}\theta \tag{46}$$

Simplifying and stacking the solutions in vector form,

$$\begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} = \vec{x}(t) = V\begin{bmatrix} \frac{1}{2LC}\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}\int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \\ \frac{1}{2LC}\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}\int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \end{bmatrix} \tag{47}$$

2.(b)iii:

We wish to show $\vec{x}(t)$ is unbounded, given some bounded input $u(t)$. When showing a vector is bounded, we can show that all of its individual, scalar entries are bounded. Alternatively, when showing a vector is unbounded, it is enough to show that one of its entries will be unbounded. Note that $i_L(t) = y_1(t) + y_2(t)$ (which we see by computing $\vec{x}(t) = V\vec{y}(t)$). We can show that this quantity is unbounded. Recall that

$$y_1(t) = \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \tag{48}$$

$$y_2(t) = \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \tag{49}$$

$$\implies i_L(t) = \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta + \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}u(\theta)\,\mathrm{d}\theta \tag{50}$$

Now, we have to make some choice of a bounded input $u(t)$ so the entire term is unbounded as $t \to \infty$. We can choose $u(t) = \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t} + \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t} = 2\cos\left(\frac{1}{\sqrt{LC}}t\right)$ which is a bounded sinusoidal function. We can first compute $i_L(t)$ with this input:

$$i_L(t) = \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta}\left(\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta} + \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}\right)\mathrm{d}\theta + \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}\left(\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}\theta} + \mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}\theta}\right)\mathrm{d}\theta \tag{51}$$

$$= \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t 1 + \mathrm{e}^{-\mathrm{j}\frac{2}{\sqrt{LC}}\theta}\,\mathrm{d}\theta + \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\int_0^t 1 + \mathrm{e}^{\mathrm{j}\frac{2}{\sqrt{LC}}\theta}\,\mathrm{d}\theta \tag{52}$$

$$= \frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\left(t + \frac{1 - \mathrm{e}^{-\mathrm{j}\frac{2}{\sqrt{LC}}t}}{\mathrm{j}\frac{2}{\sqrt{LC}}}\right) + \frac{\mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2LC}\left(t + \frac{\mathrm{e}^{\mathrm{j}\frac{2}{\sqrt{LC}}t} - 1}{\mathrm{j}\frac{2}{\sqrt{LC}}}\right) \tag{53}$$

$$= \frac{t}{LC}\left(\frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t} + \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2}\right) + \frac{1}{\sqrt{LC}}\left(\frac{\mathrm{e}^{\mathrm{j}\frac{1}{\sqrt{LC}}t} - \mathrm{e}^{-\mathrm{j}\frac{1}{\sqrt{LC}}t}}{2\mathrm{j}}\right) \tag{54}$$

$$= \frac{t}{LC} \cos\left(\frac{t}{\sqrt{LC}}\right) + \frac{1}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right) \tag{55}$$

Notice that the cos and sin terms are bounded, but the cos term is multiplied by a t, so as $t \to \infty$, $i_L(t) \to \infty$. Hence, the system is unstable. Generally, we say a system with eigenvalues having negative real part implies stability. Here, the real part of the eigenvalues is 0, so the system is unstable.

(c) Thus far, we have dealt with continuous systems so it also makes sense to consider discrete systems. Consider the discrete system

$$x[i+1] = 2x[i] + u[i] \tag{56}$$

with $x[0] = 0$.

Is the system stable or unstable? If unstable, find a bounded input sequence $u[i]$ that causes the system to "blow up".

Solution: Notice that, if we had the system

$$x[i+1] = 2x[i] \tag{57}$$

then we can write $x[i+1] = 2^i x[1]$. So, if we can somehow make $x[1]$ nonzero using a bounded input (e.g. equal to 1, for simplicity), then as $i \to \infty$, $x[i+1] \to \infty$. We know that $x[0] = 0$, and that $x[1] = 2x[0] + u[0] = u[0]$. Hence, we can set $u[0] = 1$ and then $x[1] = 1$. We have achieved what we wanted, i.e. to make $x[1]$ a nonzero value using the bounded input $u[0] = 1$. Now, for the other timesteps $i > 0$, we can set $u[i] = 0$ since that would leave us with the system in eq. (57). Written explicitly, our bounded input is

$$u[i] = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases} \tag{58}$$

Contributors:

- Anish Muthali.
- Neelesh Ramachandran.
- Anant Sahai.
- Regina Eckert.
- Kareem Ahmad.
- Sidney Buchbinder.