1. Eigenvalue Placement in Discrete Time

Consider the following linear discrete time system

\[
\vec{x}[i + 1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + w[i]
\]  

(a) Is the system given in eq. (1) stable?

(b) Derive a state space representation of the resulting closed loop system. Use state feedback of the form:

\[
u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]
\]

Hint: If you’re having trouble parsing the expression for \(u[i]\), note that \(\begin{bmatrix} f_1 & f_2 \end{bmatrix}\) is a row vector, while \(\vec{x}[i]\) is column vector. What happens when we multiply a row vector with a column vector like this?}
(c) **Find the appropriate state feedback constants**, $f_1, f_2$, **that place the eigenvalues of the state space representation matrix at** $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

(d) **Is the system now stable in closed-loop**, using the control feedback coefficients $f_1, f_2$ that we derived above?

(e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (3). As before, we use $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \bar{x}[i]$ to try and control the system.

$$\bar{x}[i + 1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \bar{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \bar{w}[i]$$  \hspace{1cm} (3)

**What would the desired eigenvalues now be?** Can you move all the eigenvalues to where you want? In particular, can you make this system stable given the form of the input?
(f) **Practice** Can you place the eigenvalues at complex conjugates, such that \( \lambda_1 = a + jb, \lambda_2 = a - jb \) using only real feedback gains \( f_1, f_2 \)? How about placing them at any arbitrary complex numbers, such that \( \lambda_1 = a + jb, \lambda_2 = c + jd \)?

2. **Uncontrollability**

Consider the following discrete-time system with the given initial state:

\[
\vec{x}[i + 1] = \begin{bmatrix}
2 & 0 & 0 \\
-3 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix} \vec{x}[i] + \begin{bmatrix}
0 \\
0 \\
2 \\
\end{bmatrix} u[i] \\
\]

\[
\vec{x}[0] = \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
\]

(a) Is the system controllable?

(b) Is it possible to reach \( \vec{x}[\ell] = \begin{bmatrix}
-2 \\
4 \\
6 \\
\end{bmatrix} \) for some \( \ell \)? For what input sequence \( u[i] \) up to \( i = \ell - 1 \)?
(c) Is it possible to reach \( \vec{x}[\ell] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} \) for some \( \ell \)? For what input sequence \( u[i] \) up to \( i = \ell - 1 \)?

*Hint: look at the intermediate results of the previous subpart, where you wrote down what \( x[0], x[1], \) etc. were. Apply these new values to those expressions.*

(d) Find the set of all \( \vec{x}[2] \), given that you are free to choose the \( u[0] \) and \( u[1] \) of your choice.