1. System Identification by Means of Least Squares

Working through this question will help you understand better how we can use experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares techniques you learned in 16A. You will later do this in lab for your robot car.

As you were told in 16A, least-squares and its variants are not just the basic workhorses of machine learning in practice, they play a conceptually central place in our understanding of machine learning well beyond least-squares.

Throughout this question, you should consider measurements to have been taken from one long trace through time.

(a) Consider the scalar discrete-time system

\[ x[i + 1] = ax[i] + bu[i] + w[i] \]  

(1)

Where the scalar state at time \( i \) is \( x[i] \), the input applied at time \( i \) is \( u[i] \) and \( w[i] \) represents some external disturbance that also participated at time \( i \) (which we cannot predict or control, it’s a purely random disturbance).

Assume that you have measurements for the states \( x[i] \) from \( i = 0 \) to \( m \) and also measurements for the controls \( u[i] \) from \( i = 0 \) to \( m - 1 \).

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters \( a \) and \( b \).
(b) What if there were now two distinct scalar inputs to a scalar system

\[ x[i + 1] = ax[i] + b_1 u_1[i] + b_2 u_2[i] + w[i] \]  

(2)

and that we have measurements as before, but now also for both of the control inputs.

**Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters** \(a, b_1, b_2\).

(c) **What could go wrong in the previous case?** For what kind of inputs would make least-squares fail to give you the parameters you want?

(d) Now consider the two dimensional state case with a single input.

\[
\bar{x}[i + 1] = \begin{bmatrix} x_1[i + 1] \\ x_2[i + 1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \bar{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \bar{w}[i] 
\]  

(3)

**How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters** \(a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2\)? Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts).

**Hint:** What work/computation can we reuse across the two problems?
2. Stability Examples and Counterexamples

(a) Consider the circuit below with \( R = 1 \, \Omega \), \( C = 0.5 \, \text{F} \), and \( u(t) \) is some waveform bounded between \(-1\) and \(1\) (for example \( \cos(t) \)). Furthermore assume that \( v_C(0) = 0 \, \text{V} \) (that the capacitor is initially discharged).

\[
\begin{align*}
\text{\( u(t) \)} & \rightarrow \hspace{1cm} \text{\( v_C(t) \)} \\
\text{\( R \)} & \rightarrow \hspace{1cm} \text{\( C \)}
\end{align*}
\]

This circuit can be modeled by the differential equation

\[
\frac{dv_C(t)}{dt} = -2v_C(t) + 2u(t) \tag{4}
\]

Show that the differential equation is always stable (that is, as long as the input \( u(t) \) is bounded, \( v_C(t) \) also stays bounded). Consider what this means in the physical circuit.

(b) Consider the discrete system

\[
x[i + 1] = 2x[i] + u[i] \tag{5}
\]

with \( x[0] = 0 \).

Is the system stable or unstable?

If unstable, find a bounded input sequence \( u[i] \) that causes the system to “blow up”. Is there still a (non-trivial) bounded input sequence that does not cause the system to “blow up”?

(c) [Practice, but challenging:] Now, suppose that in the circuit of part (a) we replaced the resistor with an inductor, \( L = 1 \, \text{mH} \). Repeat part (a) for the new circuit (with an inductor).

Hint: You might find it useful to revisit the process of generating the state-space equations for \( v_C(t) \) and \( i_L(t) \) as done in Note 4 for the LC Tank. The difference is that here, we have an input voltage.

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