Discussion 6A

For linear algebra review, it may be helpful to refer to EECS16A Note 8 (Matrix Subspaces) and Note 9 (Eigenvalues and Eigenvectors).

1. Linear Algebra Review

For the following matrices, find the following properties:

i. **What is the column space of the matrix?**

ii. **What is the null space of the matrix?**

iii. **What are the eigenvalues and corresponding eigenspaces for the matrix?**

(a) $\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$

Solution:

i. $\mathbb{R}^2$. Inspection: the columns of the matrix cannot be written as multiples of each other, so they are linearly independent. Since these vectors are in $\mathbb{R}^2$ and there are 2 linearly independent vectors, these are enough to span $\mathbb{R}^2$, which must then be the column space.

ii. $\{\vec{0}\}$. The linear independence of the columns also implies $\{\vec{0}\}$ as the solution set of $A\vec{x} = \vec{0}$ where A is the matrix, and thus the null space is $\{\vec{0}\}$.

iii. $\lambda_1 = 2$ has the corresponding eigenspace: Span $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$\lambda_2 = 3$ has the corresponding eigenspace: Span $\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$

(b) $\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$.

Solution:

i. Span $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$. If proceeding by inspection, note that the matrix columns are multiples, so their span is equivalent to the span of one column. Note that it does not matter which column is chosen. Thus the column space can be written as the span of one column from the matrix.

ii. Span $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$. Since the columns are integer multiples of each other, a solution to $A\vec{x} = \vec{0}$ can be constructed, and then extrapolated to the null space.

iii. $\lambda_1 = -3$ has the corresponding eigenspace: Span $\left(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right)$

$\lambda_2 = 0$ has the corresponding eigenspace: Span $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

2. Changing Coordinates and Systems of Differential Equations, I

Recall from lecture that matrix differential equations follow the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) \tag{1}$$

where $\vec{x}(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Note that $\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t)$ is equivalent to $\begin{bmatrix} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}x_n(t)}{\mathrm{d}t} \end{bmatrix}$ where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$. In other words, taking the derivative of the vector is the same as taking the derivative elementwise of its components. When A is diagonal, we can treat the matrix differential equation as a system of n separate, scalar differential equations. In this discussion, we will use **change of variables** to tackle a matrix differential equation where A is not diagonal. This will help us model the behavior of more complex circuits where A will usually be non-diagonal.

First, we can practice by solving a matrix differential equation with diagonal A. Suppose we have the following differential equation (valid for $t \geq 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\vec{x}(t) \tag{2}$$

with initial condition $\vec{x}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(a) **Write the matrix differential equation as a system of individual, scalar differential equations and solve for $\vec{x}(t)$ for $t \geq 0$.**

Solution: We can rewrite the matrix differential equation as follows:

$$\begin{bmatrix} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3}$$

Simplifying the matrix-vector multiplication on the right hand side, we obtain

$$\begin{bmatrix} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -9x_1(t) \\ -2x_2(t) \end{bmatrix} \tag{4}$$

Now we can write these as individual equations:

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -9x_1(t) \tag{5}$$

$$\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -2x_2(t) \tag{6}$$

As for the initial conditions, we have

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \tag{7}$$

so $x_1(0) = -1$ and $x_2(0) = 3$. Now we have two differential equations and their initial conditions. We know that they will have solutions of the form

$$\begin{cases} x_1(t) = A_1 e^{b_1 t} \\ x_2(t) = A_2 e^{b_2 t} \end{cases} \tag{8}$$

for some A_1, A_2, b_1, and b_2 that we need to find. To find A_1, we can use the initial condition on $x_1(t)$, namely $x_1(0) = -1$:

$$x_1(0) = A_1 e^{b_1(0)} = A_1 \underset{\substack{\uparrow \\ x_1(0)=-1}}{=} -1 \tag{9}$$

Similarly, for A_2 we can use $x_2(0) = 3$:

$$x_2(0) = A_2 e^{b_2(0)} = A_2 \underset{\substack{\uparrow \\ x_2(0)=3}}{=} 3 \tag{10}$$

Next, to find b_1, we can use the differential equation for $x_1(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((-1)e^{b_1 t}\right) = (-1)b_1 e^{b_1 t} \underset{\substack{\uparrow \\ \text{eq. (5)}}}{=} -9\left((-1)e^{b_1 t}\right) \tag{11}$$

$$\implies b_1 = -9. \tag{12}$$

Similarly, we can use the differential equation for $x_2(t)$ to find b_2:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(3e^{b_1 t}\right) = 3b_2 e^{b_1 t} \underset{\substack{\uparrow \\ \text{eq. (6)}}}{=} -2\left(3e^{b_2 t}\right) \tag{13}$$

$$\implies b_2 = -2 \tag{14}$$

Combining all of these, we have the individual solutions as

$$\begin{cases} x_1(t) = -e^{-9t} \\ x_2(t) = 3e^{-2t} \end{cases} \tag{15}$$

We can combine this into a vector by setting the ith component of $\vec{x}(t)$ to be $x_i(t)$ as follows:

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} \tag{16}$$

Now, suppose we are actually interested in a different set of variables with the following differential equations:

$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} = -5y_1(t) + 2y_2(t) \tag{17}$$

$$\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} = 6y_1(t) - 6y_2(t) \tag{18}$$

(b) **Write out the above system of differential equations in matrix form**. *HINT: Define the matrix differential equation in terms of* $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$. *What is your "A" matrix here?* Can we solve this system in a similar way as we did above?

Solution: We can look at the right hand sides of eq. (17) and eq. (18). This is effectively the opposite of the initial steps of part **2**.a. We can "stack" the equations here, in that eq. (17) corresponds to the first row of the matrix differential equation and eq. (18) corresponds to the second row. Stacking these equations in vector form, we get

$$\begin{bmatrix} -5y_1(t) + 2y_2(t) \\ 6y_1(t) - 6y_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \vec{y}(t) \tag{19}$$

Looking at the left hand sides of eq. (17) and eq. (18), we can stack them in a vector and simplify as follows:

$$\begin{bmatrix} \frac{\mathrm{d}y_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}y_2(t)}{\mathrm{d}t} \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{y}(t) \tag{20}$$

Hence, our matrix differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}(t) = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A}\vec{y}(t) \tag{21}$$

Unfortunately, we cannot immediately solve this using techniques we have already learned. However, the next few parts will walk you through the process of finding a solution!

As you may have noticed, it is not possible to solve the differential equation using methods we have already covered in this class. We can try to use change of variables to turn this problem into one with a diagonal system since we know how to solve these types of equations. Consider the strategy outlined in fig. 1. We want to change variables to $\vec{\widetilde{y}}(t)$, such that we end up with a differential equation where $\widetilde{A}$ will be diagonal. This is especially important when there is no clear path to a solution with just the system involving $\vec{y}(t)$ (as is the case here).

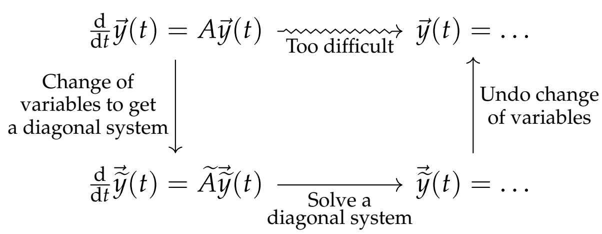

Figure 1: A Strategy to Solve for $\vec{y}(t)$

(c) We can define $\vec{\widetilde{y}}(t) = \begin{bmatrix} \widetilde{y}_1(t) \\ \widetilde{y}_2(t) \end{bmatrix}$ to achieve the goal described above. Consider the following relationship between $\vec{y}(t)$ and $\vec{\widetilde{y}}(t)$:

$$y_1(t) = -\widetilde{y}_1(t) + 2\widetilde{y}_2(t) \tag{22}$$

$$y_2(t) = 2\widetilde{y}_1(t) + 3\widetilde{y}_2(t) \tag{23}$$

(For now, take this change of variables as a given. We will explain how to come up with transformations like this a bit later.)

Write out this transformation in matrix form ($\vec{y}(t) = V\vec{\widetilde{y}}(t)$ for some V). This will give us a representation for $\vec{y}(t)$ in terms of $\vec{\widetilde{y}}(t)$. **Then, find a way to represent $\vec{\widetilde{y}}(t)$ in terms of $\vec{y}(t)$.** What conditions need to hold for this representation to work? Recall that, the inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^{-1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{24}$$

Solution: Using a similar idea of "stacking" equations as in part **2**.b, we can start by "stacking" the right hand sides of eq. (22) and eq. (23):

$$\begin{bmatrix} -\widetilde{y}_1(t) + 2\widetilde{y}_2(t) \\ 2\widetilde{y}_1(t) + 3\widetilde{y}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \widetilde{y}_1(t) \\ \widetilde{y}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\vec{\widetilde{y}}(t) \tag{25}$$

Stacking the left hand sides of eq. (22) and eq. (23) yields $\vec{y}(t)$. Combining the stacked left hand sides and right hand sides, we have

$$\vec{y}(t) = \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}_{V} \vec{\tilde{y}}(t) \tag{26}$$

Now, we want a representation for $\vec{\tilde{y}}(t)$ in terms of $\vec{y}(t)$. We need V to be invertible for this to be the case. We can check that $\det(V) = -3 - 4 = -7 \neq 0$, which means that it is full rank and hence invertible (note, V is a square matrix). We can compute V^{-1} as follows:

$$V^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = -\frac{1}{7}\begin{bmatrix} 3 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \tag{27}$$

Thus,

$$\vec{\tilde{y}}(t) = V^{-1}\vec{y}(t) \tag{28}$$

(d) Suppose that the following initial conditions are given: $\vec{y}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$. Now that we changed variables, we also have to appropriately change our initial condition for the new variables we defined. **How do these initial conditions for $\vec{y}(t)$ translate into the initial conditions for $\vec{\tilde{y}}(t)$?** *HINT: Use the representation for $\vec{\tilde{y}}(t)$ in terms of $\vec{y}(t)$ from part 2.c.*

Solution: We know that, from eq. (28), $\vec{\tilde{y}}(t) = V^{-1}\vec{y}(t)$ for all t. This means we can plug in $t = 0$ to find $\vec{\tilde{y}}(0)$:

$$\vec{\tilde{y}}(0) = V^{-1}\vec{y}(0) = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \tag{29}$$

(e) Now, we are ready to write a new differential equation for $\vec{\tilde{y}}(t)$. **Incorporate $\vec{\tilde{y}}(t)$, your new variable, into the matrix differential equation from part 2.b to come up with a differential equation for $\vec{\tilde{y}}(t)$.** *HINT: How can we substitute the $\vec{y}(t)$ terms with terms involving $\vec{\tilde{y}}(t)$? Also, recall that, since the derivative operator is linear, we can write $\frac{\mathrm{d}}{\mathrm{d}t}M\vec{x}(t) = M\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t)$ where M is a matrix of constants.* Can we solve this system of differential equations?

Solution: The original differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y}(t) = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\vec{y}(t) \tag{30}$$

We can find a differential equation for $\vec{\tilde{y}}(t)$ by first substituting $\vec{y}(t) = V\vec{\tilde{y}}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}V\vec{\tilde{y}}(t) = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}V\vec{\tilde{y}}(t) \tag{31}$$

Using the hint, we know that $\frac{\mathrm{d}}{\mathrm{d}t}V\vec{\tilde{y}}(t) = V\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{y}}(t)$ since V is a matrix of constants. Thus,

$$V\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{y}}(t) = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}V\vec{\tilde{y}}(t) \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{y}}(t) = V^{-1}\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} V\vec{\tilde{y}}(t) \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{y}}(t) = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\vec{\tilde{y}}(t) \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{y}}(t) = \underbrace{\begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}}_{\widetilde{A}}\vec{\tilde{y}}(t) \tag{35}$$

The system is exactly like the one in part **2**.a! We know how to solve this (diagonal) matrix differential equation. We started with a seemingly complex matrix differential equation, and through the power of change of variables, we now know how to solve it. We have completed the first part of our strategy, which is highlighted in red in fig. 2.

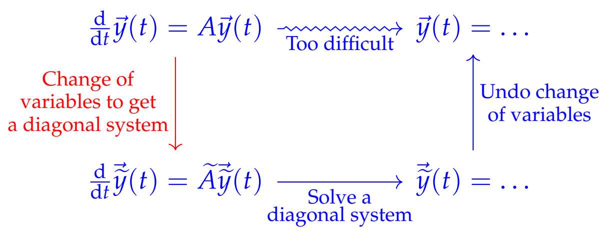

Figure 2: A Strategy to Solve for $\vec{y}(t)$

(f) **Solve the differential equation for $\vec{\tilde{y}}(t)$. Then, "undo" the change of variables from the previous parts to find a solution for $\vec{y}(t)$.** *HINT: How can we "recover" $\vec{y}(t)$ from $\vec{\tilde{y}}(t)$?* **Then, fill in the strategy diagram in fig. 3 with the following:**

(i) Mathematically, how did we define our change of variables?

(ii) In terms of $\vec{\tilde{y}}(t)$, A, and V, what matrix differential equation did we solve?

(iii) Mathematically, how did we "undo" our change of variables?

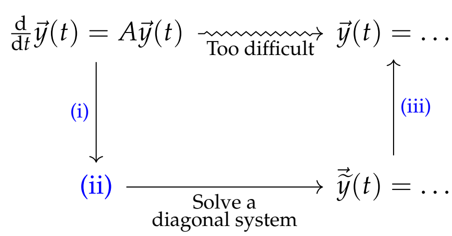

Figure 3: Mathematical Description of Our Strategy to Solve for $\vec{y}(t)$

Solution: Our differential equation for $\vec{\tilde{y}}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{y}}(t) = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\vec{\tilde{y}}(t) \tag{36}$$

We solved this exact system in part **2**.a, so we will apply that solution here. This means that

$$\vec{\tilde{y}}(t) = \begin{bmatrix} -\mathrm{e}^{-9t} \\ 3\mathrm{e}^{-2t} \end{bmatrix} \tag{37}$$

This completes the second part of our strategy, which is highlighted in red in fig. 4.

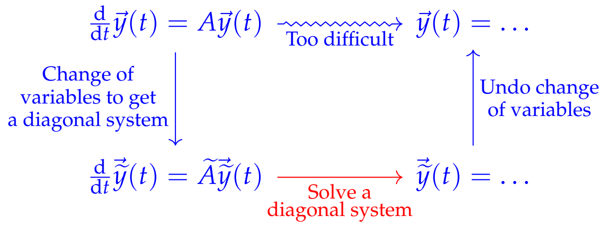

Figure 4: A Strategy to Solve for $\vec{y}(t)$

Now, we can use the representation for $\vec{y}(t)$ in terms of $\vec{\tilde{y}}(t)$ in eq. (26):

$$\vec{y}(t) = V\vec{\tilde{y}}(t) = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} \tag{38}$$

$$\vec{y}(t) = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix} \tag{39}$$

This completes the last part of our strategy, which is highlighted in red in fig. 5.

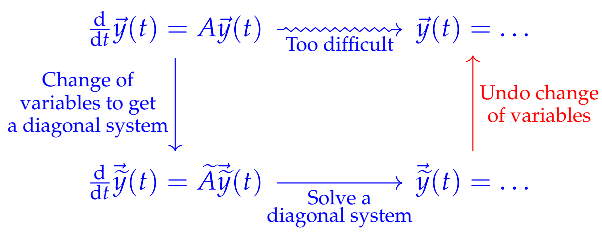

Figure 5: A Strategy to Solve for $\vec{y}(t)$

We used V to accomplish the strategy we outlined in fig. 1. We defined $\vec{\tilde{y}}(t) = V^{-1}\vec{y}(t)$ and came up with our new differential equation for $\vec{\tilde{y}}(t)$ (which we knew how to solve). This helped us come up with the solution for $\vec{y}(t)$. Incorporating the specifics of this problem and the matrices we used, our final strategy is described in fig. 6.

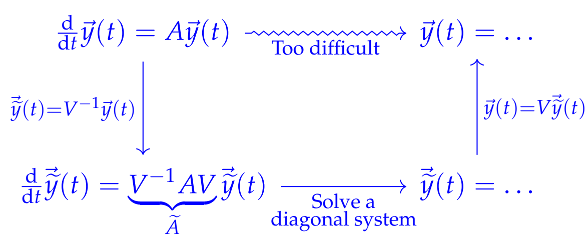

Figure 6: Mathematical Description of Our Strategy to Solve for $\vec{y}(t)$

Contributors:

- Anish Muthali.
- Anant Sahai.
- Regina Eckert.

- Nathan Lambert.
- Kareem Ahmad.
- Neelesh Ramachandran.

