In this discussion, we’re going to cover linear algebra concepts (Change of Basis, Diagonalization) that unlock powerful circuit analysis techniques going forward. We also introduce a new kind of circuit element called the “inductor”; Note 3B will be useful.

1. Coordinate Change of Basis: Examples

Consider a vector in the standard basis,

$$\vec{x} = a\vec{e}_1 + b\vec{e}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = I\vec{x}$$

(1)

where, $a,b$ are $\vec{x}$’s coordinates in the standard basis and $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the elementary standard basis vectors.

Given a new set of basis vectors, $V = \{\vec{v}_1, \vec{v}_2\}$, if $\vec{x} \in \text{span}\{V\}$, then we can find new coordinates in terms of this new basis. The new coordinates are called $a_v, b_v$ and are described,

$$\vec{x} = a_v\vec{v}_1 + b_v\vec{v}_2 = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix} = V\vec{x}_v$$

(2)

Now consider another set of basis vectors, $\mathcal{U} = \{\vec{u}_1, \vec{u}_2\}$. If $\vec{x} \in \text{span}\{\mathcal{U}\}$, then we can find the coordinates of $\vec{x}$ in terms of this basis. These coordinates are called $a_u, b_u$ and are described,

$$\vec{x} = a_u\vec{u}_1 + b_u\vec{u}_2 = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = U\vec{x}_u$$

(3)

All of these bases are equivalent representations of any vector $\vec{x} \in \mathbb{R}^2$; each with their own set of coordinates. The same logic can, of course, be extended to any number of dimensions.

$$\vec{x} = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix}$$

(4)

$$\vec{x} = I\vec{x} = V\vec{x}_v = U\vec{x}_u$$

(5)

Now that we’ve seen a conceptual overview of the change-of-basis, we can proceed with the worksheet problems.
(a) *Transformation From Standard Basis To Another Basis in \( \mathbb{R}^3 \)*

Calculate the coordinate transformation between the following bases:

\[
U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},
\]

i.e. find a matrix \( T \), such that \( \vec{x}_v = T \vec{x}_u \) where \( \vec{x}_u \) contains the coordinates of a vector in a basis of the columns of \( U \) and \( \vec{x}_v \) is the coordinates of the same vector in the basis of the columns of \( V \).

Let \( \vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) and compute \( \vec{x}_v \). Repeat this for \( \vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \). Now let \( \vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \). What is \( \vec{x}_v \)?
(b) *Transformation Between Two Bases in $\mathbb{R}^3$*

Calculate the coordinate transformation between the following bases:

\[
V = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad W = \begin{bmatrix}
0 & 0 & 1 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{bmatrix},
\]

i.e. find a matrix $T$, such that $\vec{x}_w = T \vec{x}_v$. Let $\vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_w$. Repeat this for $\vec{x}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{x}_v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{x}_w$?
2. Diagonalization

(a) Consider a matrix $A$, a matrix $V$ whose columns are the eigenvectors of $A$, and a diagonal matrix $\Lambda$ with the eigenvalues of $A$ on the diagonal (in the same order as the eigenvectors (or columns) of $V$). From these definitions, show that

$$AV = VA$$
3. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage changes as a function of the derivative of the current across it. That is:

$$V_L(t) = L \frac{dI_L(t)}{dt}$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the equivalent "fundamental" circuit for an inductor:

![Figure 1: Inductor in series with a voltage source.](image)

(a) What is the current through an inductor as a function of time? If the inductance is $L = 3H$, what is the current at $t = 6s$? Assume that the voltage source turns from 0V to 5V at time $t = 0s$, and there's no current flowing in the circuit before the voltage source turns on.

(b) Now, we add some resistance in series with the inductor, as in Figure 2.

![Figure 2: Inductor in series with a voltage source.](image)

Solve for the current $I_L(t)$ in the circuit over time, in terms of $R, L, V_S, t$. 

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(c) (Practice) Suppose $R = 500\Omega, L = 1\text{mH}, V_S = 5\text{V}$. Plot the current through and voltage across the inductor $(I_L(t), V_L(t))$, as these quantities evolve over time.
4. Fibonacci Sequence

(a) The Fibonacci sequence is built as follows: the $n$-th number ($F_n$) is sum of the previous two numbers in the sequence. That is:

$$F_n = F_{n-1} + F_{n-2}$$

If the sequence is initialized with $F_1 = 0$ and $F_2 = 1$, then the first 11 numbers in the Fibonacci sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

We can express this computation as a matrix multiplication:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = A \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

What is $A$?

(b) Find the eigenvalues and corresponding eigenvectors of $A$. 
(c) Diagonalize $A$ (that is, in the expression $A = VAV^{-1}$, solve for each component matrix.)

(d) Use the diagonalized result to show that we can arrive at an analytical result for any $F_n$:

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

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