Discussion 4A

The following sections of the textbook are useful for this discussion: 5.4, 5.5

1. Steady-State AC Node-Voltage Analysis (Hambley Example 5.6)

For this problem, assume you are told that the circuit in Figure 1 is in steady state.

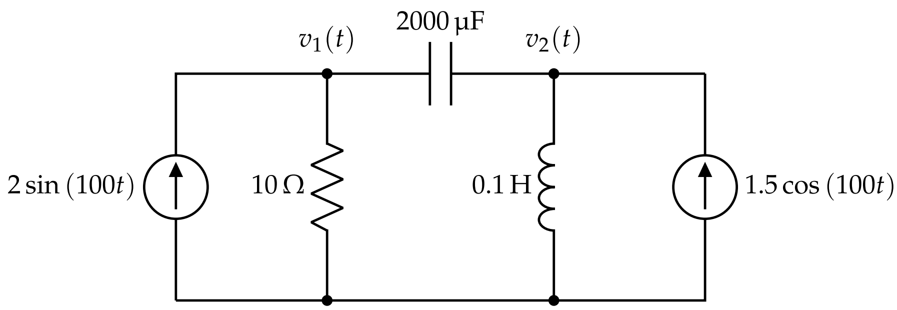


Figure 1

(a) **Redraw the given circuit in the phasor domain.** In other words, label the passive components with their impedances and sources with their phasor representation.

Solution: Recall the impedance equations:

$$Z_R = R = 10\,\Omega \tag{1}$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j(100)(2000 \times 10^{-6})} = -j5\,\Omega \tag{2}$$

$$Z_L = j\omega L = j(100)(0.1) = j10\,\Omega \tag{3}$$

Using these we can draw the simplified equation in Figure 2.

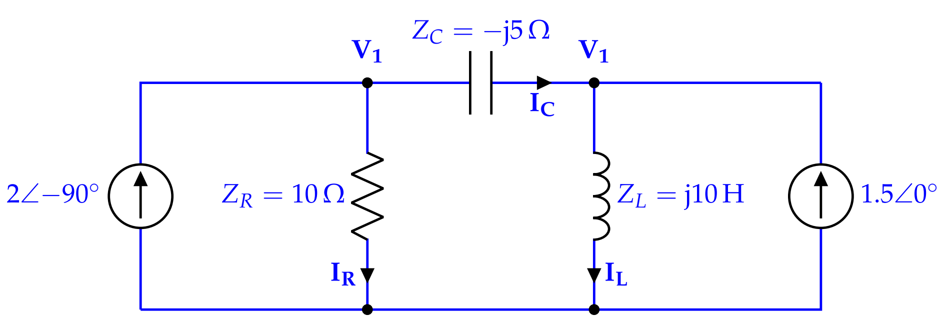


Figure 2: Transformed Circuit

(b) **Set up a system of equations for $\mathbf{V_1}$ and $\mathbf{V_2}$ using node-voltage analysis.**

Solution: We start by writing the KCL equations at the nodes $\mathbf{V_1}$ and $\mathbf{V_2}$. Let's label our currents and element voltages as shown below:

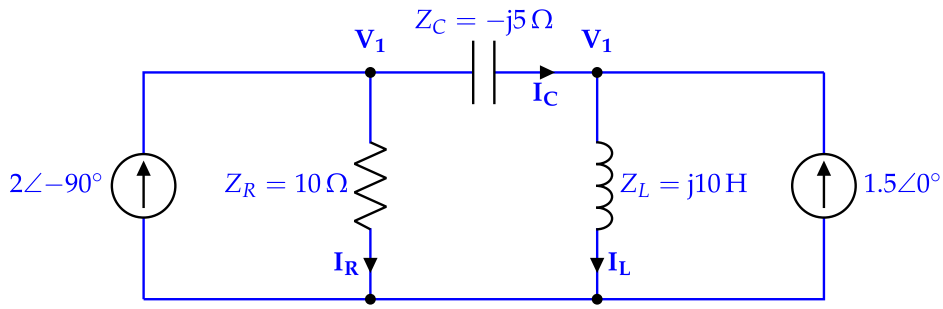


Figure 3: Transformed Circuit with Labeled Currents

Recall that it doesn't matter how we label the currents as long as passive sign convention is followed for the element voltages.

Taking KCL at the $\mathbf{V_1}$ node and plugging in element equations we get:

$$2\angle -90^\circ = \mathbf{I_R} + \mathbf{I_C} \tag{4}$$

$$2\angle -90^\circ = \frac{\mathbf{V_R}}{Z_R} + \frac{\mathbf{V_C}}{Z_C} \tag{5}$$

$$2\angle -90^\circ = \frac{\mathbf{V_1}}{10} - \frac{\mathbf{V_1} - \mathbf{V_2}}{j5} \tag{6}$$

Now, if we do the same for the $\mathbf{V_2}$ node with the same labeling, we get:

$$\mathbf{I_C} + 1.5\angle 0^\circ = \mathbf{I_R} \tag{7}$$

$$1.5\angle 0^\circ = \frac{\mathbf{V_L}}{Z_L} - \frac{\mathbf{V_C}}{Z_C} \tag{8}$$

$$1.5\angle 0^\circ = \frac{\mathbf{V_2}}{j10} - \frac{\mathbf{V_1} - \mathbf{V_2}}{-j5} \tag{9}$$

$$1.5\angle 0^\circ = \frac{\mathbf{V_2}}{j10} + \frac{\mathbf{V_1} - \mathbf{V_2}}{j5} \tag{10}$$

Thus, our systems of equations involving $\mathbf{V_1}$ and $\mathbf{V_2}$ is:

$$2\angle -90^\circ = \frac{\mathbf{V_1}}{10} - \frac{\mathbf{V_1} - \mathbf{V_2}}{j5} \tag{11}$$

$$1.5\angle 0^\circ = \frac{\mathbf{V_2}}{j10} + \frac{\mathbf{V_1} - \mathbf{V_2}}{j5} \tag{12}$$

(c) **Using the systems of equations from part (b), solve for $v_1(t)$ in steady state.**

Solution: There are many ways to go about solving the linear system of equations. Putting the equations from above in standard form, we get:

$$(0.1 + j0.2)\mathbf{V_1} - j0.2\mathbf{V_2} = -j2 \tag{13}$$

$$-j0.2\mathbf{V_1} + j0.1\mathbf{V_2} = 1.5 \tag{14}$$

Rearranging the second equation: we have that:

$$j0.1\mathbf{V_2} = 1.5 + j0.2\mathbf{V_1} \tag{15}$$

$$\mathbf{V_2} = -\text{j}15 + 2\mathbf{V_1} \tag{16}$$

Now, we can substitute $\mathbf{V_2}$ into our first equation and solve for V_1:

$$(0.1 + \text{j}0.2)\mathbf{V_1} - \text{j}0.2\mathbf{V_2} = -\text{j}2 \tag{17}$$

$$(0.1 + \text{j}0.2)\mathbf{V_1} - \text{j}0.2(-\text{j}15 + 2\mathbf{V_1}) = -\text{j}2 \tag{18}$$

$$(0.1 + \text{j}0.2)\mathbf{V_1} - 3 - \text{j}0.4\mathbf{V_1} = -\text{j}2 \tag{19}$$

$$(0.1 - \text{j}0.2)\mathbf{V_1} = 3 - \text{j}2 \tag{20}$$

$$\mathbf{V_1} = \frac{3 - \text{j}2}{0.1 - \text{j}0.2} \tag{21}$$

$$\mathbf{V_1} = \frac{\sqrt{13}\angle 146.31^\circ}{\sqrt{0.05}\angle 116.56^\circ} \tag{22}$$

$$\mathbf{V_1} = 16.1\angle 29.7^\circ \tag{23}$$

Converting our phasor to a time function, we get

$$v_1(t) = 16.1\cos(100t + 29.7^\circ) \tag{24}$$

Notice that by converting our circuit into the phasor domain, the system becomes linear! We can then solve the linear system of equations and the convert back from phasor domain to the time domain.

2. AC Power Calculations (Hambley Example 5.7)

Suppose you are given the circuit in Figure 4, where the phasor for current $i(t)$ is calculated to be $0.1414\angle 135^\circ$.

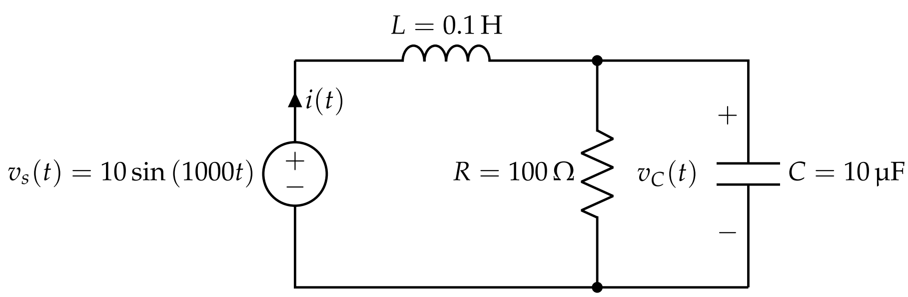


Figure 4: RLC Circuit

(a) **Compute the power and reactive power taken from the source in the circuit provided in Figure 4.**

Solution: To find the power and reactive power for the source, we have to find the power angle given by the equation:

$$\theta = \theta_v - \theta_i \tag{25}$$

where θ_v is the phase of the voltage and θ_i is the phase of the current. The angle of the source voltage is $\theta_v = -90^\circ$ and the angle of the current delivered by the source is $\theta_i = -135^\circ$. Therefore, we have:

$$\theta = -90^\circ - (-135^\circ) = 45^\circ \tag{26}$$

The rms source voltage and current are

$$V_{s,rms} = \frac{|\mathbf{V_s}|}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7.071\,\text{V} \tag{27}$$

$$I_{rms} = \frac{|\mathbf{I}|}{\sqrt{2}} = \frac{0.1414}{\sqrt{2}} = 0.1\,\text{A} \tag{28}$$

Now, using the following equations,

$$P = V_{s,rms} I_{rms} \cos(\theta) \tag{29}$$

$$Q = V_{s,rms} I_{rms} \sin(\theta) \tag{30}$$

we can solve for the power and reactive power.

$$P = 7.071 \times 0.1 \cos(45^\circ) = 0.5\,\text{W} \tag{31}$$

$$Q = 7.071 \times 0.1 \sin(45^\circ) = 0.5\,\text{VAR} \tag{32}$$

Alternate Approach:

Another way to compute P and Q is to find complex power and then take the real and imaginary parts:

$$\mathbf{S} = \frac{1}{2}\mathbf{V_s}\mathbf{I}^* \tag{33}$$

$$= \frac{1}{2}(10\angle{-90^\circ})(0.1414\angle 135^\circ) \tag{34}$$

$$= 0.707\angle 45^\circ \tag{35}$$

$$= 0.5 + \text{j}0.5 \tag{36}$$

Then, we have:

$$P = \text{Re}\{S\} = 0.5\,\text{W} \tag{37}$$

$$Q = \text{Im}\{S\} = 0.5\ \text{VAR} \tag{38}$$

(b) **Compute the power and reactive power delivered to each element in the circuit.** Assume you are given the computed currents in Figure 5.

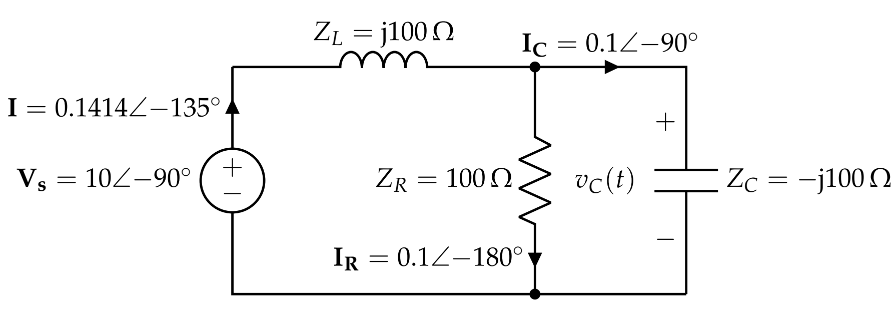


Figure 5: RLC Circuit

Solution: For the reactive power delivered to the inductor, we have:

$$Q_L = I_{\text{rms}}^2 X_L \tag{39}$$

$$= (0.1)^2(100) \tag{40}$$
$$= 1.0 \text{ VAR} \tag{41}$$

Similar for the capacitor we have:

$$Q_C = I_{c,\text{rms}}^2 X_C \tag{42}$$
$$= \left(\frac{0.1}{\sqrt{2}}\right)^2 (-100) \tag{43}$$
$$= -0.5 \text{ VAR} \tag{44}$$

Notice that the reactance X_C of the capacitance is negative. As expected, the reactive power is negative for a capacitance. The reactive power for the resistance is zero. To verify our work, let's check that the reaction power delivered by the source is equal to the sum of the reactive powers absorbed by the inductance and capacitance:

$$Q = Q_L + Q_C \tag{45}$$

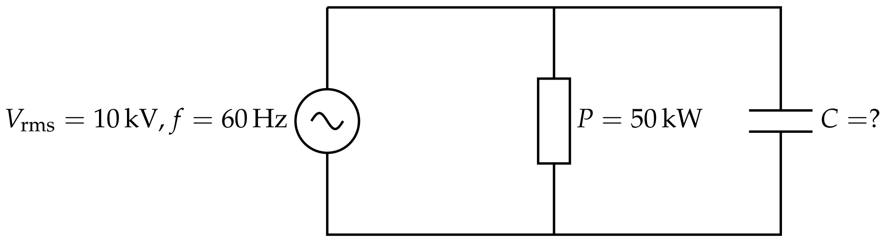
We have that the power delivered to the resistance

$$P_R = I_{R,\text{rms}}^2 R \tag{46}$$
$$= \left(\frac{|\mathbf{I_R}|}{\sqrt{2}}\right)^2 R \tag{47}$$
$$= \left(\frac{0.1}{\sqrt{2}}\right)^2 100 \tag{48}$$
$$= 0.5\,\text{W} \tag{49}$$

The power absorbed by the capacitance and inductance is given by $P_L = 0$ and $P_C = 0$. Therefore, all of the power delivered by the source is absorbed by the resistance.

3. Power-Factor Correction (Hambley Example 5.9)

A 50 kW load operates from a 60 Hz 10 kV-rms line with a power factor of 60 percent lagging.



Compute the capacitance that must be placed in parallel with the load to achieve a 90 percent power factor.

Solution: First, we need to find the load power angle:

$$\theta_L = \arccos(0.6) = 53.13^\circ \tag{50}$$

Then, we use the power-triangle concept to find the reactive power of the load.

$$Q_L = P_L \tan(\theta_L) = 66.67 \text{ kVAR} \tag{51}$$

After adding the capacitor, the power will still be 50 kW and the power angle will become:

$$\theta_{\text{new}} = \arccos(0.9) = 25.84^\circ \tag{52}$$

The new value of the reactive power will be:

$$Q_{\text{new}} = P_L \tan(\theta_{\text{new}}) = 24.22 \text{ kVAR} \tag{53}$$

Now, we find that the reactive power of the capacitance must be:

$$X_C = -\frac{V_{\text{rms}}^2}{Q_C} = \frac{(10^4)^2}{42,450} = -2356\,\Omega \tag{54}$$

Finally, since the angular frequency is $\omega = 2\pi 60 = 377$, the required capacitance will be:

$$C = \frac{1}{\omega |X_C|} = \frac{1}{377 \times 2356} = 1.126\,\mu\text{F} \tag{55}$$

