Questions

1. DFT

In order to get practice with calculating the Discrete Fourier Transform (DFT), this problem will have you calculate the DFT for a few variations on a cosine signal.

Consider a sampled signal that is a function of discrete time $x[t]$. We can represent it as a vector of discrete samples over time $\vec{x}$, of length $N$.

$$\vec{x} = \begin{bmatrix} x[0] & \ldots & x[N-1] \end{bmatrix}^T$$

(1)

Let $\vec{X} = \begin{bmatrix} X[0] & \ldots & X[N-1] \end{bmatrix}^T$ be the signal $\vec{x}$ represented in the frequency domain, then

$$\vec{x} = U \vec{X}$$

(2)

and the inverse operation is given by

$$\vec{X} = U^{-1} \vec{x} = U^* \vec{x}$$

(3)

where the columns of $U$ are the orthonormal DFT basis vectors.

$$U = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & e^{2\pi i N} & e^{2\pi i (2) N} & \ldots & e^{2\pi i (N-1) N} \\ 1 & e^{2\pi i (2) N} & e^{2\pi i (4) N} & \ldots & e^{2\pi i (2(N-1)) N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{2\pi i (N-1) N} & e^{2\pi i (2(N-1)) N} & \ldots & e^{2\pi i (N(N-1)) N} \end{bmatrix}$$

(4)

$$= \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_N & \omega_N^{2} & \ldots & \omega_N^{(N-1)} \\ 1 & \omega_N^{2} & \omega_N^{2^2} & \ldots & \omega_N^{(N-1)^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \ldots & \omega_N^{(N-1)(N-1)} \end{bmatrix}$$

(5)

where $\omega = e^{2\pi i N}$ is the $N$th primitive root of unity.

We sometimes call the components of $\vec{X}$ the DFT coefficients of the time-domain signal $\vec{x}$. We can think of the components of $\vec{X}$ as weights that represent $\vec{x}$ in the DFT basis.
(a) Let’s begin by looking at the DFT of \( x_1[n] = \cos\left(\frac{2\pi}{5}n\right) \) for \( N = 5 \) samples \( n \in \{0, 1, \ldots, 4\} \).

**Answer:** \( N = 5 \) so \( \omega_5 = e^{j\frac{2\pi}{5}} \). Plugging this into the form of \( U \) we get:

\[
U = \frac{1}{\sqrt{5}} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & e^{j\frac{4\pi}{5}} & e^{j\frac{8\pi}{5}} & e^{j\frac{12\pi}{5}} & e^{j\frac{16\pi}{5}} \\
1 & e^{j\frac{4\pi}{5}} & e^{j\frac{8\pi}{5}} & e^{j\frac{12\pi}{5}} & e^{j\frac{16\pi}{5}} \\
1 & e^{j\frac{6\pi}{5}} & e^{j\frac{12\pi}{5}} & e^{j\frac{18\pi}{5}} & e^{j\frac{24\pi}{5}} \\
1 & e^{j\frac{8\pi}{5}} & e^{j\frac{16\pi}{5}} & e^{j\frac{24\pi}{5}} & e^{j\frac{32\pi}{5}}
\end{bmatrix}
\]  

(6)

Note that DFT basis vectors take the form:

\[
u_i[n] = \frac{1}{\sqrt{5}} \cdot e^{j\frac{2i\pi}{5}n}.
\]  

(7)

(b) Write out \( \vec{x}_1 \) in terms of the DFT basis vectors. **Answer:**

\[
\cos\left(\frac{2\pi}{5}n\right) = \frac{1}{2} \left( e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n} \right)
\]  

(8)

\[
= \frac{1}{2} \left( e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n+2\pi} \right)
\]  

(9)

\[
= \sqrt{\frac{5}{2}} \left( \frac{1}{\sqrt{5}} e^{j\frac{2\pi}{5}n} + \frac{1}{\sqrt{5}} e^{j\frac{8\pi}{5}n} \right)
\]  

(10)

\[
= \sqrt{\frac{5}{2}} (u_1[n] + u_4[n])
\]  

(11)

\[
\vec{x}_1 = \sqrt{\frac{5}{2}} \vec{u}_1 + \sqrt{\frac{5}{2}} \vec{u}_4.
\]  

(12)

Note that since \( e^{j\theta} \) is periodic, \( e^{j\theta} = e^{j(\theta+2\pi)} \).

(c) Find the DFT coefficients \( X_1[k] \). **Answer:**

\[
\vec{X}_1 = U^* \vec{x}_1
\]  

(13)

\[
= \begin{bmatrix}
\tilde{u}_0 \\
\tilde{u}_1 \\
\tilde{u}_2 \\
\tilde{u}_3 \\
\tilde{u}_4
\end{bmatrix}
\]  

(14)

\[
= \sqrt{\frac{5}{2}} \begin{bmatrix}
0 \\
\sqrt{\frac{5}{2}} \\
0 \\
m_3 \\
\sqrt{\frac{5}{2}}
\end{bmatrix}
\]  

(15)

(d) Plot the time domain representation of \( x_1[n] \). Plot the magnitude, \( |X_1[k]| \), and plot the phase, \( \angle X_1[k] \), for the DFT representation \( \vec{X}_1 \).

**Answer:**
(e) Now let’s consider the case were have a non-zero phase. Let $x_2[n] = \cos\left(\frac{4\pi}{5} n + \pi\right)$. Find the DFT coefficients $\vec{X}_2$ for $\vec{x}_2$. **Answer:** Writing out $\vec{x}_2$ in terms of the DFT basis vectors we get:
\[
\cos \left( \frac{4\pi}{5} n + \pi \right) = \frac{1}{2} \left( e^{j\frac{4\pi}{5} n} e^{j\pi} + e^{-j\frac{4\pi}{5} n} e^{-j\pi} \right) 
\]

\[
= \frac{\sqrt{5}}{2} e^{j\pi} u_2[n] + \frac{\sqrt{5}}{2} e^{-j\pi} u_3[n] 
\]

\[
\hat{x}_2 = \frac{\sqrt{5}}{2} e^{j\pi} \tilde{u}_2 + \frac{\sqrt{5}}{2} e^{-j\pi} \tilde{u}_3. 
\]

Transforming \( \hat{x}_2 \) by \( U^* \) into the DFT basis then gives our DFT coefficients:

\[
\hat{X}_1 = U^* \hat{x} 
\]

\[
= \begin{bmatrix}
-\tilde{u}_0^* \\
-\tilde{u}_1^* \\
-\tilde{u}_2^* \\
-\tilde{u}_3^* \\
-\tilde{u}_4^* \\
\end{bmatrix}
\]

\[
= \frac{\sqrt{5}}{2} e^{j\pi} \tilde{u}_2 + \frac{\sqrt{5}}{2} e^{-j\pi} \tilde{u}_3 
\]

\[
= \begin{bmatrix}
0 \\
0 \\
\frac{\sqrt{5}}{2} e^{j\pi} \\
\frac{\sqrt{5}}{2} e^{-j\pi} \\
0 \\
\end{bmatrix} 
\]

\[
(f) \text{ Plot the time domain representation of } x_2[n]. \text{ Plot the magnitude, } |X_2[k]|, \text{ and plot the phase, } \angle X_2[k], \text{ for the DFT representation } \hat{X}_2. \text{ Answer:}
\]

![Time Domain Plot]

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(g) Now let’s look at the reverse direction. Given \( \vec{X}_3 = \left[ \begin{array}{cccc} 2 & e^{-j\pi/2} & 0 & 0 \end{array} \right] \), find \( x_3[n] \).

\textbf{Answer:} To convert from the DFT basis back into the standard basis, we apply \( U \).

\[ \vec{x}_3 = U \vec{X}_3 \] \hspace{1cm} (22)

\[ \begin{align*}
\vec{x}_3 &= \left[ \vec{u}_0 \quad \vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3 \quad \vec{u}_4 \right] \begin{bmatrix} 2 & e^{-j\pi/2} \\ e^{-j\pi/2} & 0 & e^{j\pi/2} \end{bmatrix} \\
&= 2\vec{u}_0 + e^{-j\pi/2} \vec{u}_1 + e^{j\pi/2} \vec{u}_4
\end{align*} \] \hspace{1cm} (23)

\[ x_3[n] = \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} e^{-j\pi/2} e^{j\frac{2\pi}{5}n} + \frac{1}{\sqrt{5}} e^{j\pi/2} e^{-j\frac{2\pi}{5}n} \] \hspace{1cm} (24)

\[ x_3[n] = \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} \cos \left( \frac{2\pi}{5} n - \frac{\pi}{2} \right) \] \hspace{1cm} (25)

\[ x_3[n] = \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} \cos \left( \frac{2\pi}{5} n - \frac{\pi}{2} \right) \] \hspace{1cm} (26)
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