EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet Discussion 14B

Questions

1. DFT

In order to get practice with calculating the Discrete Fourier Transform (DFT), this problem will have you calculate the DFT for a few variations on a cosine signal.

Consider a sampled signal that is a function of discrete time $x[t]$. We can represent it as a vector of discrete samples over time $\vec{x}$, of length N.

$$\vec{x} = \begin{bmatrix} x[0] & \dots & x[N-1] \end{bmatrix}^T \tag{1}$$

Let $\vec{X} = \begin{bmatrix} X[0] & \dots & X[N-1] \end{bmatrix}^T$ be the signal $\vec{x}$ represented in the frequency domain, then

$$\vec{x} = U\vec{X} \tag{2}$$

and the inverse operation is given by

$$\vec{X} = U^{-1}\vec{x} = U^*\vec{x} \tag{3}$$

where the columns of U are the orthonormal DFT basis vectors.

$$U = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & e^{j\frac{2\pi(2)}{N}} & \cdots & e^{j\frac{2\pi(N-1)}{N}} \\ 1 & e^{j\frac{2\pi(2)}{N}} & e^{j\frac{2\pi(4)}{N}} & \cdots & e^{j\frac{2\pi 2(N-1)}{N}} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{j\frac{2\pi(N-1)}{N}} & e^{j\frac{2\pi 2(N-1)}{N}} & \cdots & e^{j\frac{2\pi(N-1)(N-1)}{N}} \end{bmatrix} \tag{4}$$

$$= \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^1 & \omega_N^2 & \cdots & \omega_N^{(N-1)} \\ 1 & \omega_N^2 & \omega_N^{2\cdot 2} & \cdots & \omega_N^{(N-1)2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)(N-1)} \end{bmatrix}, \tag{5}$$

where $\omega_N = e^{j\frac{2\pi}{N}}$ is the Nth primitive root of unity.

We sometimes call the components of $\vec{X}$ the *DFT coefficients* of the time-domain signal $\vec{x}$. We can think of the components of $\vec{X}$ as weights that represent $\vec{x}$ in the DFT basis.

(a) Let's begin by looking at the DFT of $x_1[n] = \cos\left(\frac{2\pi}{5}n\right)$ for $N = 5$ samples $n \in \{0, 1, \ldots, 4\}$. Compute the DFT basis matrix U.

Answer: $N = 5$ so $\omega_5 = e^{j\frac{2\pi}{5}}$. Plugging this into the form of U we get:

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{5}} & e^{j\frac{4\pi}{5}} & e^{j\frac{6\pi}{5}} & e^{j\frac{8\pi}{5}} \\ 1 & e^{j\frac{4\pi}{5}} & e^{j\frac{8\pi}{5}} & e^{j\frac{12\pi}{5}} & e^{j\frac{16\pi}{5}} \\ 1 & e^{j\frac{6\pi}{5}} & e^{j\frac{12\pi}{5}} & e^{j\frac{18\pi}{5}} & e^{j\frac{24\pi}{5}} \\ 1 & e^{j\frac{8\pi}{5}} & e^{j\frac{16\pi}{5}} & e^{j\frac{24\pi}{5}} & e^{j\frac{32\pi}{5}} \end{bmatrix} \tag{6}$$

Note that DFT basis vectors take the form:

$$u_i[n] = \frac{1}{\sqrt{5}} \cdot e^{j\frac{2i\pi}{5}n}. \tag{7}$$

(b) Write out $\vec{x}_1$ in terms of the DFT basis vectors. **Answer:**

$$\cos\left(\frac{2\pi}{5}n\right) = \frac{1}{2}\left(e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n}\right) \tag{8}$$

$$= \frac{1}{2}\left(e^{j\frac{2\pi}{5}n} + e^{-j\frac{2\pi}{5}n+2\pi}\right) \tag{9}$$

$$= \frac{\sqrt{5}}{2}\left(\frac{1}{\sqrt{5}}e^{j\frac{2\pi}{5}n} + \frac{1}{\sqrt{5}}e^{j\frac{8\pi}{5}n}\right) \tag{10}$$

$$= \frac{\sqrt{5}}{2}(u_1[n] + u_4[n]) \tag{11}$$

$$\vec{x}_1 = \frac{\sqrt{5}}{2}\vec{u}_1 + \frac{\sqrt{5}}{2}\vec{u}_4. \tag{12}$$

Note that since $e^{j\theta}$ is periodic, $e^{j\theta} = e^{j(\theta+2\pi)}$.

(c) Find the DFT coefficients $X_1[k]$. **Answer:**

$$\vec{X}_1 = U^*\vec{x}_1 \tag{13}$$

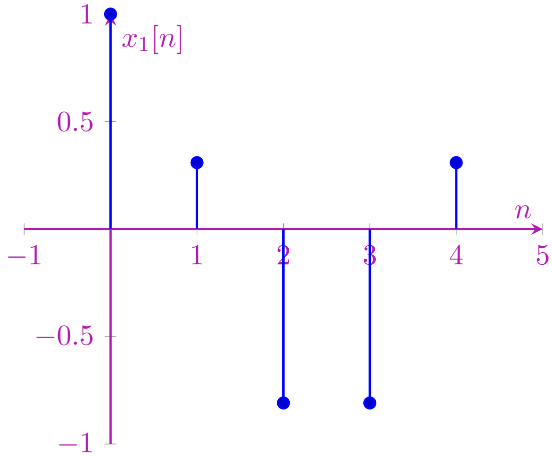
$$= \begin{bmatrix} —\vec{u}_0^*— \\ —\vec{u}_1^*— \\ —\vec{u}_2^*— \\ —\vec{u}_3^*— \\ —\vec{u}_4^*— \end{bmatrix} \frac{\sqrt{5}}{2}(\vec{u}_1 + \vec{u}_4) \tag{14}$$

$$= \begin{bmatrix} 0 \\ \frac{\sqrt{5}}{2} \\ 0 \\ 0 \\ \frac{\sqrt{5}}{2} \end{bmatrix} \tag{15}$$

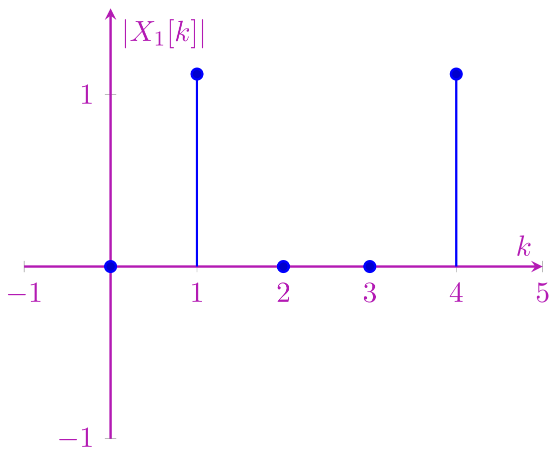
(d) Plot the time domain representation of $x_1[n]$. Plot the magnitude, $|X_1[k]|$, and plot the phase, $\angle X_1[k]$, for the DFT representation $\vec{X}_1$.

Answer:

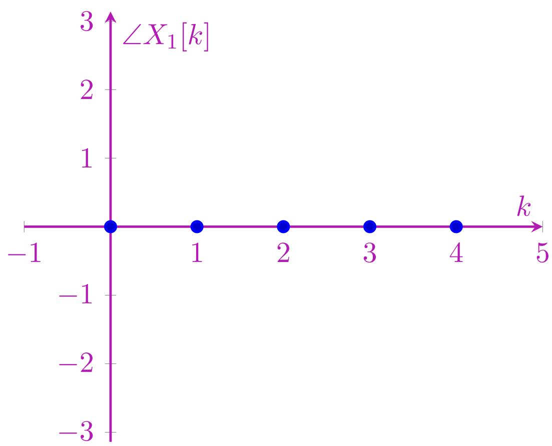
Time Domain



Frequency Domain Magnitude



Frequency Domain Phase



(e) Now let's consider the case were have a non-zero phase. Let $x_2[n] = \cos\left(\frac{4\pi}{5}n + \pi\right)$. Find the DFT coefficients $\vec{X}_2$ for $\vec{x}_2$. **Answer:** Writing out $\vec{x}_2$ in terms of the DFT basis vectors we get:

$$\cos\left(\frac{4\pi}{5}n+\pi\right) = \frac{1}{2}\left(e^{j\frac{4\pi}{5}n}e^{j\pi} + e^{-j\frac{4\pi}{5}n}e^{-j\pi}\right) \tag{16}$$

$$= \frac{\sqrt{5}}{2}e^{j\pi}u_2[n] + \frac{\sqrt{5}}{2}e^{-j\pi}u_3[n]) \tag{17}$$

$$\vec{x}_2 = \frac{\sqrt{5}}{2}e^{j\pi}\vec{u}_2 + \frac{\sqrt{5}}{2}e^{-j\pi}\vec{u}_3. \tag{18}$$

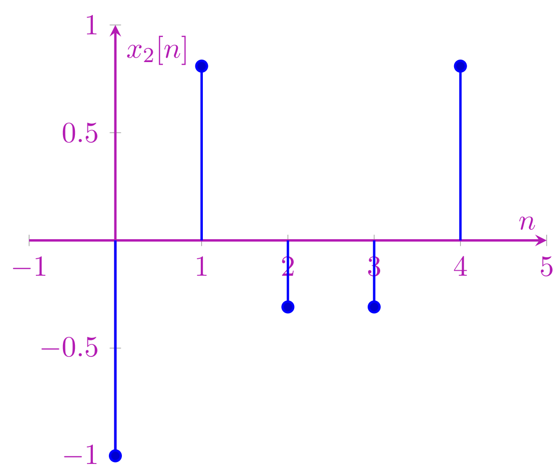
Transforming $\vec{x}_2$ by $U*$ into the DFT basis then gives our DFT coefficients:

$$\vec{X}_1 = U^*\vec{x} \tag{19}$$

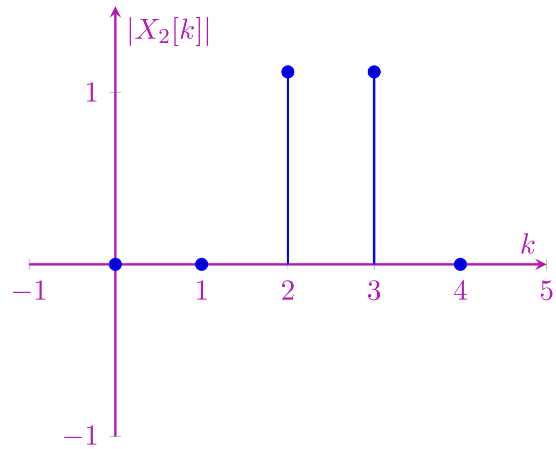
$$= \begin{bmatrix} —\vec{u}_0^*— \\ —\vec{u}_1^*— \\ —\vec{u}_2^*— \\ —\vec{u}_3^*— \\ —\vec{u}_4^*— \end{bmatrix} \left(\frac{\sqrt{5}}{2}e^{j\pi}\vec{u}_2 + \frac{\sqrt{5}}{2}e^{-j\pi}\vec{u}_3\right) \tag{20}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{5}}{2}e^{j\pi} \\ \frac{\sqrt{5}}{2}e^{-j\pi} \\ 0 \end{bmatrix} \tag{21}$$

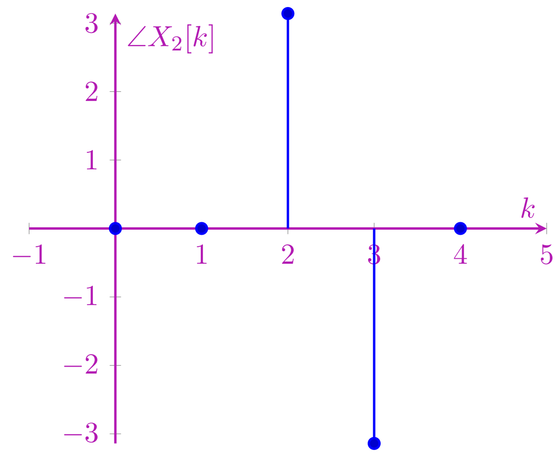
(f) Plot the time domain representation of $x_2[n]$. Plot the magnitude, $|X_2[k]|$, and plot the phase, $\angle X_2[k]$, for the DFT representation $\vec{X}_2$. **Answer:**



Frequency Domain Magnitude



Frequency Domain Phase



(g) Now let's look at the reverse direction. Given $\vec{X}_3 = \begin{bmatrix} 2 & e^{-j\frac{\pi}{2}} & 0 & 0 & e^{j\frac{\pi}{2}} \end{bmatrix}^\top$, find $x_3[n]$.

Answer: To convert from the DFT basis back into the stardard basis, we apply U.

$$\vec{x}_3 = U\vec{X}_3 \tag{22}$$

$$= \begin{bmatrix} \vec{u}_0 & \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix} \begin{bmatrix} 2 \\ e^{-j\frac{\pi}{2}} \\ 0 \\ 0 \\ e^{j\frac{\pi}{2}} \end{bmatrix} \tag{23}$$

$$= 2\vec{u}_0 + e^{-j\frac{\pi}{2}}\vec{u}_1 + e^{j\frac{\pi}{2}}\vec{u}_4 \tag{24}$$

$$x_3[n] = \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}e^{-j\frac{\pi}{2}}e^{j\frac{2\pi}{5}n} + \frac{1}{\sqrt{5}}e^{j\frac{\pi}{2}}e^{-j\frac{2\pi}{5}n} \tag{25}$$

$$= \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}\cos\left(\frac{2\pi}{5}n - \frac{\pi}{2}\right). \tag{26}$$

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