EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet

Discussion 14A

1. Complex Inner Products, Projections, and Orthonormality

To understand how we want to define complex inner products, it is useful to first recall how we came to define real inner products. Inner products grow out of our desire to do projections. Projections themselves are intimately connected to the idea of orthogonality.

When projecting a vector $\vec{v}$ onto another vector $\vec{u}$, the result needs to be $r\vec{u}$ where r is some constant. In other words, we want a vector that is linearly dependent with $\vec{u}$ so that it captures all of $\vec{v}$ that is in the direction of $\vec{u}$.

Because the idea of *direction* is so important, we can first focus on the distilled embodiments of directions themselves — namely unit vectors. Vectors whose length is 1 essentially are all about direction since their magnitude/length is known.

If we have a vector $\vec{u} \in \mathbb{R}^n$, recall that we can define the projection operator as $P_{\vec{u}} = \frac{\vec{u}\vec{u}^\top}{\|\vec{u}\|^2}$ that acts on vectors. This means, for any vector $\vec{v}$, we can project $\vec{v}$ onto the vector $\vec{u}$ by computing the projection $P_{\vec{u}}\vec{v}$[1]:

$$P_{\vec{u}}\vec{v} = \frac{\vec{u}\vec{u}^\top}{\|\vec{u}\|^2}\vec{v} = \frac{\vec{u}^\top\vec{v}}{\|\vec{u}\|^2}\vec{u}. \tag{1}$$

This gives us a vector that is in the direction of $\vec{u}$ and is a multiple $\vec{u}^\top\vec{v}$ of $\vec{u}$. This is a projection because the residual $\vec{v} - P_{\vec{u}}\vec{v}$ is orthogonal to $\vec{u}$:

$$\vec{u}^\top(\vec{v} - P_{\vec{u}}\vec{v}) = \vec{u}^\top\left(\vec{v} - \frac{\vec{u}^\top\vec{v}}{\|\vec{u}\|^2}\vec{u}\right) = \vec{u}^\top\vec{v} - \frac{\left(\vec{u}^\top\vec{u}\right)\left(\vec{u}^\top\vec{v}\right)}{\|\vec{u}\|^2} = \vec{u}^\top\vec{v} - \vec{u}^\top\vec{v} = 0. \tag{2}$$

(a) First, we want to consider formulating the projection with real inner products instead of norms. Recall that we defined the real inner product as $\langle\vec{u},\vec{v}\rangle = \sum_i u_i v_i = \vec{u}^\top\vec{v} = \vec{v}^\top\vec{u}$. Use this to rewrite the matrix $P_{\vec{u}}$ just using inner products instead of norms.

Use this to show that when we project a vector $\vec{v}$ onto a vector $\vec{u}$, that the result is $\frac{\langle\vec{v},\vec{u}\rangle}{\langle\vec{u},\vec{u}\rangle}\vec{u}$.

Answer: Using that $\|\vec{u}\|^2 = \langle\vec{u},\vec{u}\rangle$,

$$P_{\vec{u}} = \frac{\vec{u}\vec{u}^\top}{\|\vec{u}\|^2} = \vec{u}\frac{\vec{u}^\top}{\langle\vec{u},\vec{u}\rangle} \tag{3}$$

Then

$$P_{\vec{u}}\vec{v} = \vec{u}\frac{\vec{u}^\top}{\langle\vec{u},\vec{u}\rangle}\vec{v} = \vec{u}\left(\frac{\vec{u}^\top\vec{v}}{\langle\vec{u},\vec{u}\rangle}\right) = \vec{u}\frac{\langle\vec{v},\vec{u}\rangle}{\langle\vec{u},\vec{u}\rangle} = \frac{\langle\vec{v},\vec{u}\rangle}{\langle\vec{u},\vec{u}\rangle}\vec{u}. \tag{4}$$

[1] For example, in the case of least squares, this projection would be onto $\mathrm{col}(A)$ and be of the form $P_{\mathrm{col}(A)}(\vec{b}) = A\vec{\hat{x}} = A(A^\top A)^{-1}A^\top\vec{b}$.

(b) Now, let's expand this to the case where the vectors are complex.

To be precise, an n-dimensional *complex vector* is just like an n-dimensional vector that we're used to working with, *but* each of the components are complex numbers instead of real numbers.

One-dimensional real vectors are just 1×1 vectors of the form $\begin{bmatrix} r \end{bmatrix}$, for $r \in \mathbb{R}$. Correspondingly, one-dimensional complex vectors are just vectors of the form $\begin{bmatrix} c \end{bmatrix}$, for $c \in \mathbb{C}$. Make sure to keep in mind the difference between these vectors, and the familiar real/complex numbers.

If $\vec{v}$ is a complex vector, we define its length, or norm, by the equality

$$\|\vec{v}\|^2 = \sum_{i=1}^{n} |v_i|^2 = \sum_{i=1}^{n} v_i \overline{v_i}. \tag{5}$$

Here, recall that $\overline{v_i}$ is the complex conjugate of the complex number v_i. Notice that this is similar *but not exactly equal* to the norm on real vectors defined by $\|\vec{v}\|^2 = \sum_{i=1}^{n} v_i^2$; in particular, $v_i \overline{v_i} \neq v_i^2$, unless v_i is real.

A *complex unit vector* is a complex vector with norm 1, exactly analogous to real vectors with norm 1 being the "regular" unit vectors we're used to working with.

The one-dimensional real unit vectors are just $\begin{bmatrix} 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \end{bmatrix}$, since $|1|^2 = |-1|^2 = 1$. What are the one-dimensional complex unit vectors?

Answer: The vectors to consider are $\begin{bmatrix} e^{j\theta} \end{bmatrix}$ for all $\theta \in [0, 2\pi)$. There are infinitely many such vectors.

(c) If we are considering *real* vectors, then two vectors $\vec{v}_1$ and $\vec{v}_2$ are *linearly dependent* if $\vec{v}_1 \in \operatorname{span}(\vec{v}_2)$. That is, there is a real scalar $r \in \mathbb{R}$ such that $\vec{v}_1 = r\vec{v}_2$. In this case, r is the coefficient of the projection of $\vec{v}_1$ onto $\vec{v}_2$, and in particular $r = \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}$.

Now let's consider the case of *complex* vectors. Two complex vectors $\vec{v}_1$ and $\vec{v}_2$ are *linearly dependent* if there is a complex scalar $c \in \mathbb{C}$ such that $\vec{v}_1 = c\vec{v}_2$. In this way, even though we haven't defined a way to project general complex vectors onto other complex vectors yet, we already know the projection coefficient c [2] of $\vec{v}_1$ onto $\vec{v}_2$. (Remember, the only reason we can do this is because they're linearly dependent, a very special case).

The previous problem had us pick out that the one-dimensional complex unit vectors are $\begin{bmatrix} e^{j\theta} \end{bmatrix}$ for $0 \leq \theta < 2\pi$. Two special cases are $\begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} e^{j \cdot 0} \end{bmatrix}$ and $\begin{bmatrix} j \end{bmatrix} = \begin{bmatrix} e^{j \cdot \frac{\pi}{2}} \end{bmatrix}$. Calculate the projection coefficient of the projection of $\begin{bmatrix} 1 \end{bmatrix}$ onto $\begin{bmatrix} j \end{bmatrix}$ and for the projection of $\begin{bmatrix} j \end{bmatrix}$ onto $\begin{bmatrix} 1 \end{bmatrix}$.

Answer: Let's call these coefficients of projection c_1 and c_2. We have

$$\begin{bmatrix} 1 \end{bmatrix} = c_1 \begin{bmatrix} j \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} j \end{bmatrix} = c_2 \begin{bmatrix} 1 \end{bmatrix} \tag{7}$$

From this we can directly solve to see that $c_1 = -j$, $c_2 = j$. But dividing the first equation by c_1 shows that $\frac{1}{c_1} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} j \end{bmatrix}$, while the second equation says $c_2 \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} j \end{bmatrix}$. Thus $c_1 = \frac{1}{c_2}$, so they're reciprocals. In the case of *real* vectors, for a unit vector $\vec{v}$, the only unit vectors that are linearly dependent with $\vec{v}$ are $\vec{v}$ and $-\vec{v}$. The projections of $\vec{v}$ onto $-\vec{v}$ and $-\vec{v}$ onto $\vec{v}$ both have coefficient $r = -1$. This happens because -1 is its own reciprocal.

[2] Remember, this is not the projection! This coefficient is specifically a scalar value – projections are vector-valued.

But in the complex case, there are many unit vectors that are linearly dependent with $\vec{v}$ – just multiply $\vec{v}$ by $e^{j\theta}$ for any θ to get another unit vector. The reciprocal of a complex scalar with magnitude 1 (not norm – remember, these are *scalars*) – is the complex conjugate of the scalar. So, if $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent complex unit vectors with $\vec{v}_1 = c_1\vec{v}_2$ and $\vec{v}_2 = c_2\vec{v}_1$, then $c_1 = \overline{c_2}$.

(d) The previous example has shown you that when complex numbers are involved, the order matters of who is being projected onto whom, even when both are unit vectors. Now that we do not have symmetry, we need to be more careful in formulating a projection operator for complex vectors.

Consider a complex vector $\vec{u}$. Define the following operator: $P_{\vec{u}} = \frac{\vec{u}\vec{u}^*}{\|\vec{u}\|^2}$. Here $\vec{u}^* = \left(\overline{\vec{u}}\right)^\top$ is the conjugate transpose of the vector $\vec{u}$ – in other words, we take the complex conjugate of every entry in $\vec{u}$, and take the transpose of the result. For real vectors $\vec{u}$, this is the same as the projection operator we had above because the complex conjugate would do nothing.

We propose that it's the projection operator in the complex case, that is, $P_{\vec{u}}\vec{v}$ is the projection of $\vec{v}$ onto the span of $\vec{u}$. We don't know for sure this is the case – we haven't proved it, only guessed it, after all – but we plan to show it a little later, once we do an example.

In the 1-d case, suppose $\vec{u} = \begin{bmatrix} c \end{bmatrix}$ is a complex unit vector. Then

$$P_{\vec{u}} = \frac{\vec{u}\vec{u}^*}{\|\vec{u}\|^2} = \frac{1}{|c|^2}\begin{bmatrix} c \end{bmatrix}\begin{bmatrix} \overline{c} \end{bmatrix} = \frac{1}{|c|^2}\begin{bmatrix} c\overline{c} \end{bmatrix} = \frac{1}{|c|^2}\begin{bmatrix} |c|^2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} = I_1. \tag{8}$$

Now let's check the 2-d case. As in the 1-d case, we want our vectors to be linearly dependent at first, to check our intuition. For two unit vectors $\vec{v}_1 = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}$, compute the projection of $\vec{v}_1$ onto $\vec{v}_2$ using $P_{\vec{v}_2}$ and the projection of $\vec{v}_2$ onto $\vec{v}_1$ using $P_{\vec{v}_1}$. What are the projection coefficients c_1 and c_2 such that $P_{\vec{v}_2}\vec{v}_1 = c_1\vec{v}_2$, and $P_{\vec{v}_1}\vec{v}_2 = c_2\vec{v}_1$? Do these make sense? How are they related to each other?

Answer: We start with the projection operator for unit vectors $\vec{u}$. Since $\|\vec{u}\| = 1$,

$$P_{\vec{u}} = \frac{\vec{u}\vec{u}^*}{\|\vec{u}\|^2} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\begin{bmatrix} \overline{u}_1 & \overline{u}_2 \end{bmatrix} = \begin{bmatrix} u_1\overline{u_1} & u_1\overline{u_2} \\ u_2\overline{u_1} & u_2\overline{u_2} \end{bmatrix}. \tag{9}$$

Let's compute $P_{\vec{v}_1}$ and $P_{\vec{v}_2}$:

$$P_{\vec{v}_1} = \begin{bmatrix} v_{11}\overline{v_{11}} & v_{11}\overline{v_{12}} \\ v_{12}\overline{v_{11}} & v_{12}\overline{v_{12}} \end{bmatrix} = \begin{bmatrix} \frac{j}{\sqrt{2}} \cdot \overline{\frac{j}{\sqrt{2}}} & \frac{j}{\sqrt{2}} \cdot \overline{-\frac{1}{\sqrt{2}}} \\ -\frac{1}{\sqrt{2}} \cdot \overline{\frac{j}{\sqrt{2}}} & -\frac{1}{\sqrt{2}} \cdot \overline{-\frac{1}{\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} \frac{j}{\sqrt{2}} \cdot -\frac{j}{\sqrt{2}} & \frac{j}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \cdot -\frac{j}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}. \tag{11}$$

$$P_{\vec{v}_2} = \begin{bmatrix} v_{21}\overline{v_{21}} & v_{21}\overline{v_{22}} \\ v_{22}\overline{v_{21}} & v_{22}\overline{v_{22}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot \overline{\frac{1}{\sqrt{2}}} & \frac{1}{\sqrt{2}} \cdot \overline{\frac{j}{\sqrt{2}}} \\ \frac{j}{\sqrt{2}} \cdot \overline{\frac{1}{\sqrt{2}}} & \frac{j}{\sqrt{2}} \cdot \overline{\frac{j}{\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \cdot -\frac{j}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \cdot -\frac{j}{\sqrt{2}} \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}. \tag{13}$$

Now for the projections,

$$P_{\vec{v}_1}\vec{v}_2 = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} = \vec{v}_2 \tag{15}$$

$$P_{\vec{v}_2}\vec{v}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \vec{v}_1 \tag{17}$$

Notice that in this case, the answers make perfect sense. The two vectors are linearly dependent to each other and so the projection operator returns the vector being projected.

For the projection coefficients, we have

$$P_{\vec{v}_2}\vec{v}_1 = \vec{v}_1 = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = c_1 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} = c_1\vec{v}_2 \tag{18}$$

$$P_{\vec{v}_1}\vec{v}_2 = \vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} = c_2 \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = c_2\vec{v}_1 \tag{19}$$

and obtain $c_1 = j$, $c_2 = -j$. In this case $c_1 = \overline{c_2}$, in agreement with the previous part.

(e) We have everything we need to develop a generic inner product for complex vectors. Given $\vec{u}$ a complex vector, we would like the projection of $\vec{v}$ onto $\vec{u}$ to have coefficient $\frac{\langle \vec{v},\vec{u}\rangle}{\langle \vec{u},\vec{u}\rangle}$ (so that the projection is $P_{\vec{u}}\vec{v} = \frac{\langle \vec{v},\vec{u}\rangle}{\langle \vec{u},\vec{u}\rangle}\vec{u}$). Furthermore, from the previous parts, we know that the projection coefficient of the projection of $\vec{v}$ onto $\vec{u}$ is the conjugate of the projection coefficient of the projection of $\vec{u}$ onto $\vec{v}$, so we would like $\langle \vec{v},\vec{u}\rangle = \overline{\langle \vec{u},\vec{v}\rangle}$.

Verify that the inner product $\langle \vec{v},\vec{u}\rangle = \vec{u}^*\vec{v}$ satisfies these properties. (Note the switching of the order of the arguments – $\vec{v}$ is first in the inner product notation but second in the conjugate transpose notation!)

Answer: We would like to show that $P_{\vec{u}}\vec{v} = \frac{\langle \vec{v},\vec{u}\rangle}{\langle \vec{u},\vec{u}\rangle}\vec{u}$, which we can do via a calculation:

$$P_{\vec{u}}\vec{v} = \frac{\vec{u}\vec{u}^*}{\|\vec{u}\|^2}\vec{v} = \vec{u}\frac{\vec{u}^*\vec{v}}{\|\vec{u}\|^2} = \frac{\langle \vec{v},\vec{u}\rangle}{\langle \vec{u},\vec{u}\rangle}\vec{u}. \tag{20}$$

Note here that we used the calculation

$$\|\vec{u}\|^2 = \sum_{i=1}^{n} u_i\overline{u}_i = \left(\overline{\vec{u}}\right)^\top \vec{u} = \vec{u}^*\vec{u} = \langle \vec{u},\vec{u}\rangle. \tag{21}$$

The second property can be verified by a shorter argument:

$$\langle \vec{v},\vec{u}\rangle = \vec{u}^*\vec{v} = \overline{\vec{v}^*\vec{u}} = \overline{\langle \vec{u},\vec{v}\rangle}. \tag{22}$$

In the case of $\vec{u}$, $\vec{v}$ being real vectors, then $\langle \vec{v}, \vec{u} \rangle = \vec{u}^\top \vec{v}$, showing that this inner product reduces to the inner product of real vectors.

One thing to note is that we could have defined the equally valid inner product $\langle \vec{v}, \vec{u} \rangle = \vec{v}^* \vec{u}$. The choice we make is due to popular convention, but importantly *we are going to stick with our choice* of $\langle \vec{v}, \vec{u} \rangle = \vec{u}^* \vec{v}$. Mixing up the inner product order would mess up whatever calculation we're doing.

Anyways, we now have a proposed notion of projection and inner product. Let's make sure they work together and are self-consistent.

(f) Recall the standard definitions of *orthogonal* and *orthonormal*. Two vectors $\vec{u}$ and $\vec{v}$ are defined to be *orthogonal* when $\langle \vec{u}, \vec{v} \rangle = 0$. Note that in the complex inner product case, when $\langle \vec{u}, \vec{v} \rangle = 0$, $\langle \vec{v}, \vec{u} \rangle = 0$ as well, since $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$. Two vectors $\vec{u}$ and $\vec{v}$ are *orthonormal* if they are orthogonal and $\|\vec{u}\| = \|\vec{v}\| = 1$.

Let $\vec{u}$ be a complex vector. Verify the basic Pythagorean property that the residual $\vec{r} = \vec{v} - P_{\vec{u}}\vec{v}$ that remains after projecting $\vec{v}$ onto $\vec{u}$ is orthogonal to $\vec{u}$.

Answer: We want to show that when projecting a vector $\vec{v}$ onto $\vec{u}$, the residual is orthogonal to $\vec{u}$. Specifically, the residual $\vec{r}$ is

$$\vec{r} = \vec{v} - P_{\vec{u}}\vec{v}. \tag{23}$$

To check orthogonality, we need to directly compute $\langle \vec{r}, \vec{u} \rangle$ and see that it equals zero.

$$\begin{aligned}
\langle \vec{r}, \vec{u} \rangle &= \langle \vec{v} - P_{\vec{u}}\vec{v}, \vec{u} \rangle && (24)\\
&= \vec{u}^* \left(\vec{v} - P_{\vec{u}}\vec{v}\right) && (25)\\
&= \vec{u}^* \left(\vec{v} - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u}\right) && (26)\\
&= \vec{u}^*\vec{v} - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u}^*\vec{u} && (27)\\
&= \langle \vec{v}, \vec{u} \rangle - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}\langle \vec{u}, \vec{u} \rangle && (28)\\
&= \langle \vec{v}, \vec{u} \rangle - \langle \vec{v}, \vec{u} \rangle && (29)\\
&= 0. && (30)
\end{aligned}$$

This establishes orthogonality with respect to our definition.

(g) One of the main reasons we study complex inner products in this class is to be able to handle the DFT, which is coming up in the next section. This involves the manipulation of a particular matrix which has orthonormal columns.

Show that if the columns of a square matrix are orthonormal, then the conjugate transpose of the matrix is its inverse. (*Hint: show for a matrix M with orthonormal columns, that $M^*M = I$*).

Answer: Consider the matrix

$$M = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}. \tag{31}$$

Then we would have

$$M^*M = \begin{bmatrix} \vec{v}_1^* \\ \vdots \\ \vec{v}_n^* \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \tag{32}$$

$$= \begin{bmatrix} \vec{v}_1^*\vec{v}_1 & \cdots & \vec{v}_1^*\vec{v}_n \\ \vdots & \ddots & \vdots \\ \vec{v}_n^*\vec{v}_1 & \cdots & \vec{v}_n^*\vec{v}_n \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} \langle \vec{v}_1, \vec{v}_1 \rangle & \cdots & \langle \vec{v}_n, \vec{v}_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \vec{v}_1, \vec{v}_n \rangle & \cdots & \langle \vec{v}_n, \vec{v}_n \rangle \end{bmatrix} \tag{34}$$

$$= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I_n \tag{35}$$

where the last line is from the definitions of orthogonal and orthonormal in the complex case.

The remainder of the worksheet is several examples of complex projections, each with a different type of projection coefficient.

(h) Let's consider two vectors that aren't linearly dependent. For the two unit vectors $\vec{v}_1 = \begin{bmatrix} j \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}$, compute the projection of $\vec{v}_1$ onto $\vec{v}_2$ and the projection of $\vec{v}_2$ onto $\vec{v}_1$, using either the projection operator or the inner product. What are the coefficients of the projection? Do these make sense? How are they related to each other?

Answer:

Let's compute $P_{\vec{v}_1}$ and $P_{\vec{v}_2}$.

$$P_{\vec{v}_1} = \vec{v}_1\vec{v}_1^* = \begin{bmatrix} j \\ 0 \end{bmatrix} \begin{bmatrix} -j & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{36}$$

$$P_{\vec{v}_2} = \vec{v}_2\vec{v}_2^* = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{j}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}. \tag{37}$$

For projections, we can now calculate:

$$P_{\vec{v}_1}\vec{v}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \tag{38}$$

$$P_{\vec{v}_2}\vec{v}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} j \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{j}{2} \\ -\frac{1}{2} \end{bmatrix} \tag{39}$$

The coefficients can be obtained via inner product:

$$P_{\vec{v}_1}\vec{v}_2 = \langle \vec{v}_2, \vec{v}_1 \rangle \vec{v}_1 = (\vec{v}_1^* \vec{v}_2)\vec{v}_1 = -\frac{j}{\sqrt{2}}\vec{v}_1 \tag{40}$$

$$P_{\vec{v}_2}\vec{v}_1 = \langle \vec{v}_1, \vec{v}_2 \rangle \vec{v}_2 = (\vec{v}_2^* \vec{v}_1)\vec{v}_2 = \frac{j}{\sqrt{2}}\vec{v}_2. \tag{41}$$

Once again, we see the complex conjugate relationship. Because the original vectors were not linearly dependent with each other, the "reciprocal" relationship doesn't exist at the level of these coefficients. The spirit of it remains in this complex conjugacy.

(i) Here we have another example with complex vectors where the two vectors are linearly dependent, but the vectors are now linearly dependent by a coefficient that is no longer purely imaginary. Specifically, we have the following two vectors $\vec{v}_1 = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} \frac{-1+j}{2} \\ \frac{-1-j}{2} \end{bmatrix}$.

Compute the projection of $\vec{v}_1$ onto $\vec{v}_2$ and the projection of $\vec{v}_2$ onto $\vec{v}_1$ using either the projection operator or the inner product. What are the projection coefficients? Do these make sense? How are they related to each other?

Answer: Let's compute $P_{\vec{v}_1}$ and $P_{\vec{v}_2}$. For $P_{\vec{v}_1}$,

$$P_{\vec{v}_1} = \vec{v}_1 \vec{v}_1^* = \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{j}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{42}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}. \tag{43}$$

For $P_{\vec{v}_2}$,

$$P_{\vec{v}_2} = \vec{v}_2 \vec{v}_2^* \tag{44}$$

$$= \vec{v}_2 \vec{v}_2^* \tag{45}$$

$$= \begin{bmatrix} \frac{-1+j}{2} \\ \frac{-1-j}{2} \end{bmatrix} \begin{bmatrix} \frac{-1-j}{2} & \frac{-1+j}{2} \end{bmatrix} \tag{46}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}. \tag{47}$$

Now for the projections,

$$P_{\vec{v}_1}\vec{v}_2 = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{-1+j}{2} \\ \frac{-1-j}{2} \end{bmatrix} \tag{48}$$

$$= \begin{bmatrix} \frac{-1+j}{2} \\ \frac{-1-j}{2} \end{bmatrix} = \vec{v}_2 \tag{49}$$

$$P_{\vec{v}_2}\vec{v}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{50}$$

$$= \begin{bmatrix} \frac{j}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \vec{v}_1. \tag{51}$$

Notice that in this case, the answers make sense again the two vectors are linearly dependent, so the projection operator returns the original vector. However, it's less clean than the unit vector case. The two vectors are linearly dependent to each other, but they are not linearly dependent only with an imaginary coefficient.

Here's the solution with the inner product (that also gives us the projection coefficients):

$$P_{\vec{v}_1}\vec{v}_2 = \langle \vec{v}_2, \vec{v}_1 \rangle \vec{v}_1 = (\vec{v}_1^* \vec{v}_2)\vec{v}_1 = \frac{1+j}{\sqrt{2}}\vec{v}_1 \tag{52}$$

$$P_{\vec{v}_2}\vec{v}_1 = \langle \vec{v}_1, \vec{v}_2 \rangle \vec{v}_2 = (\vec{v}_2^* \vec{v}_1)\vec{v}_2 = \frac{1-j}{\sqrt{2}}\vec{v}_2. \tag{53}$$

Notice how the coefficients $\frac{1\pm j}{\sqrt{2}}$ are conjugates of each other, a pattern we've observed throughout the worksheet.

(j) Now let's see how to extend this for non-unit vectors. For two vectors $\vec{v}_1 = \begin{bmatrix} j \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}$, compute the projection of $\vec{v}_1$ onto $\vec{v}_2$ and the projection of $\vec{v}_2$ onto $\vec{v}_1$. What are the projection coefficients? Do these make sense? How are they related to each other?

Answer: We calculate $P_{\vec{v}_1}$ and $P_{\vec{v}_2}$ as usual, making sure to use the unnormalized formula.

$$\begin{aligned}
P_{\vec{v}_1} &= \frac{\vec{v}_1 \vec{v}_1^*}{\|\vec{v}_1\|^2} && (54)\\
&= \frac{\vec{v}_1 \vec{v}_1^*}{2} && (55)\\
&= \frac{1}{2}\begin{bmatrix} j \\ -1 \end{bmatrix}\begin{bmatrix} -j & -1 \end{bmatrix} && (56)\\
&= \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}. && (57)\\
P_{\vec{v}_2} &= \frac{\vec{v}_2 \vec{v}_2^*}{\|\vec{v}_2\|^2} && (58)\\
&= \frac{\vec{v}_2 \vec{v}_2^*}{2} && (59)\\
&= \frac{1}{2}\begin{bmatrix} 1 \\ j \end{bmatrix}\begin{bmatrix} 1 & -j \end{bmatrix} && (60)\\
&= \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}. && (61)
\end{aligned}$$

Hence, the projections are

$$P_{\vec{v}_1}\vec{v}_2 = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ j \end{bmatrix} = \begin{bmatrix} 1 \\ j \end{bmatrix} = \vec{v}_2 \tag{62}$$

$$P_{\vec{v}_2}\vec{v}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} j \\ -1 \end{bmatrix} = \begin{bmatrix} j \\ -1 \end{bmatrix} = \vec{v}_1. \tag{63}$$

The inner product method also gives the projection coefficients:

$$P_{\vec{v}_1}\vec{v}_2 = \frac{\langle \vec{v}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 = \frac{\vec{v}_1^* \vec{v}_2}{\vec{v}_1^* \vec{v}_1}\vec{v}_1 = -j\vec{v}_1 \tag{64}$$

$$P_{\vec{v}_2}\vec{v}_1 = \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle}\vec{v}_2 = \frac{\vec{v}_2^* \vec{v}_1}{\vec{v}_2^* \vec{v}_2}\vec{v}_2 = j\vec{v}_2 \tag{65}$$

The projection coefficients are therefore $\pm j$. Once again, we see the complex conjugate relationship.

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