EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet **Discussion 13A**

This discussion will recap a lot of the key concepts covered in lecture last week.

1. Linear Approximation

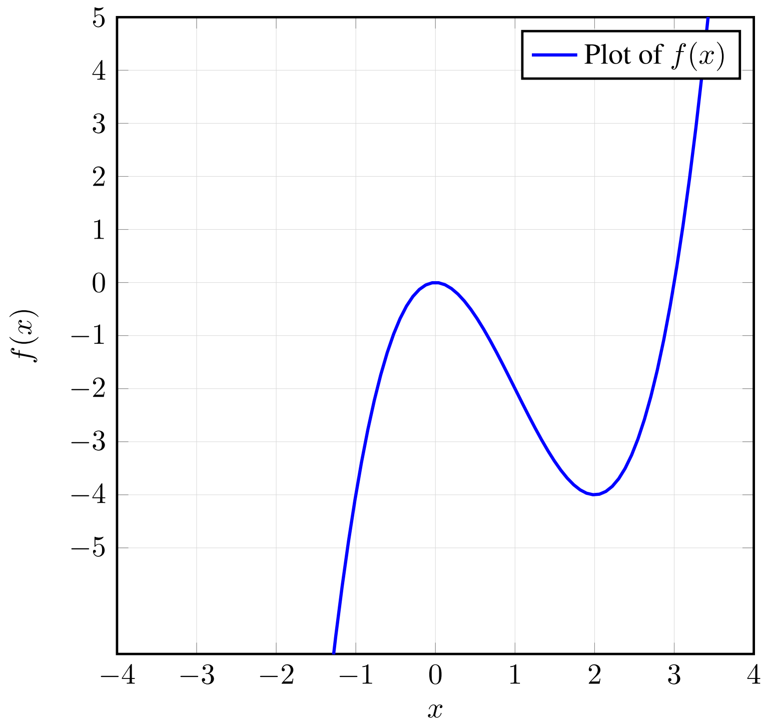
A common way to approximate a nonlinear function is to perform linearization near a point. In the case of a one-dimensional function $f(x)$, the linear approximation of $f(x)$ at a point $x_\star$ is given by

$$f(x) \approx f(x_\star) + f'(x_\star) \cdot (x - x_\star), \tag{1}$$

where $f'(x_\star) := \left.\frac{df(x)}{dx}\right|_{x=x_\star}$ is the derivative of $f(x)$ at $x = x_\star$.

Keep in mind that wherever we see $x_\star$, this denotes a *constant value* or operating point.

(a) Suppose we have the single-variable function $f(x) = x^3 - 3x^2$. We can plot the function $f(x)$ as follows:



i. Write the linear approximation of the function around an arbitrary point $x_\star$.

Answer:

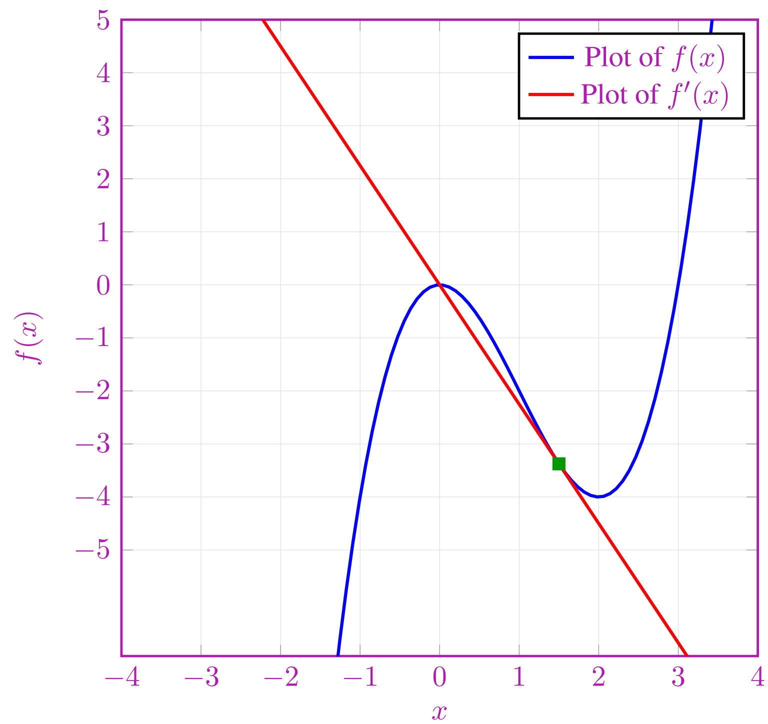
$$f(x) \approx f(x_\star) + f'(x_\star) \cdot (x - x_\star) \tag{2}$$
$$\approx f(x_\star) + (3 \cdot x_\star^2 - 6x_\star) \cdot (x - x_\star) \tag{3}$$

ii. Use the expression above to linearize the function around the point $x = 1.5$. Draw the linearization into the plot of part i).

Answer:

$$f(x) \approx f(1.5) + \left(3 \cdot 1.5^2 - 6 \cdot 1.5\right) \cdot (x - 1.5) \quad (4)$$

$$\approx -3.375 + (-2.25) \cdot (x - 1.5) \quad (5)$$



Now that we have this specific point's linearization, we understand how the function behaves around the point. Let's use this linearization to evaluate the function's approximation at $x = 1.7$ (based on our approximation at $x = 1.5$, we want to see how a $\delta = +0.2$ shift in the x value changes the corresponding $f(x)$ value). How does this approximation compare to the exact value of the function at $x = 1.7$?

$$f(1.7) \approx -3.375 + (-2.25) \cdot (1.7 - 1.5) \quad (6)$$

$$\approx -3.375 - 0.45 \quad (7)$$

$$\approx -3.825 \quad (8)$$

Comparing to the exact value $f(1.7) = 1.7^3 - 3 \cdot 1.7^2 = -3.757$, we find that the difference is 0.068. Not too bad! What if we repeat with $\delta = 1$? To do so, we must use the approximation around $x = 1.5$ to compute $x = 2.5$, and compare to the exact value $f(2.5)$. How does our new approximation compare to the exact result?

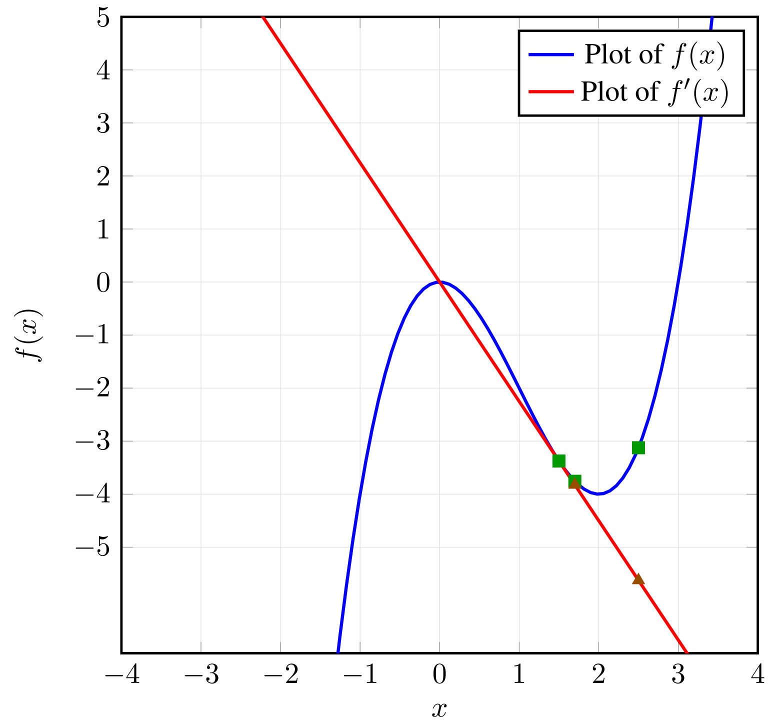
$$f(2.5) \approx -3.375 + (-2.25) \cdot (2.5 - 1.5) \quad (9)$$

$$\approx -3.375 - 2.25 \quad (10)$$

$$\approx -5.625 \quad (11)$$

Comparing to the exact value $f(2.5) = 2.5^3 - 3 \cdot 2.5^2 = -3.125$, we find that the difference is much larger; the error jumped to 2.5! This is an error multiplication of $\frac{2.5}{0.068} \approx 37$, even though our δ only multiplied by 5. What happened?

Looking at the actual function, we see that the function has a significant curvature between our "anchor point" of $x_\star = 1.5$ and $x = 2.5$. Our linear model is unable to capture this curvature, and so we estimated $f(2.5)$ as if the function kept decreasing, as it did around $x = 1.5$ (where the slope was -2.25).



Now, we can extend this to higher dimensional functions. In the case of a two-dimensional function $f(x,y)$, the linear approximation of $f(x,y)$ at a point $(x_\star, y_\star)$ is given by

$$f(x,y) \approx f(x_\star, y_\star) + f_x(x_\star, y_\star) \cdot (x - x_\star) + f_y(x_\star, y_\star) \cdot (y - y_\star). \tag{12}$$

where $f_x(x_\star, y_\star)$ is the partial derivative of $f(x,y)$ with respect to x at the point $(x_\star, y_\star)$:

$$f_x(x_\star, y_\star) = \left.\frac{\partial f(x,y)}{\partial x}\right|_{(x_\star, y_\star)} \tag{13}$$

and $f_y(x_\star, y_\star)$ is the partial derivative of $f(x,y)$ with respect to y at the point $(x_\star, y_\star)$.

(b) Now, let's see how we can derive partial derivatives. When we are given a function $f(x,y)$, we calculate the partial derivative of f with respect to x by fixing y and taking the derivative with respect to x. Given the function $f(x,y) = x^2 y$, find the partial derivatives $f_y(x,y)$ and $f_x(x,y)$.

Answer: We have

$$f_y(x,y) = x^2 \tag{14}$$

and

$$f_x(x,y) = 2xy. \tag{15}$$

(c) Write out the linear approximation of f near $(x_\star, y_\star)$.

Answer: Based on the formula in eq. (12), we can write that:

$$f(x,y) \approx f(x_\star, y_\star) + 2x_\star y_\star \cdot (x - x_\star) + x_\star^2 \cdot (y - y_\star). \tag{16}$$

(d) We want to see if the approximation arising from linearization of this function is reasonable for a point close to our point of evaluation. First, approximate $f(x,y)$ at the point $(2.01, 3.01)$ using $(x_\star, y_\star) = (2,3)$, and compare the result to $f(2.01, 3.01)$.

Answer: Let $\delta = 0.01$. Then, the true value of $f(2.01, 3.01)$ is

$$f(2.01, 3.01) = (2+\delta)^2(3+\delta) = (4 + 4\delta + \delta^2)(3+\delta) = 12 + 16\delta + 7\delta^2 + \delta^3. \tag{17}$$

On the other hand, our approximation is

$$f(2.01, 3.01) \approx f(2,3) + 2 \cdot 2 \cdot 3 \cdot \delta + 2^2 \cdot \delta = 12 + 16\delta. \tag{18}$$

As we can see, our approximation removes the terms with δ^2 and δ^3. When δ is sufficiently small, these terms become very small, and hence our approximation is reasonable.

The actual numerical values are:

$$\begin{aligned} f(2,3) &= 12 \\ f(2.01, 3.01) &\approx 12.16 \qquad \text{(using linearization)} \\ f(2.01, 3.01) &= 12.160701 \qquad \text{(exact evaluation of} f) \end{aligned}$$

(e) Suppose we have now a vector-valued function $f(\vec{x},\vec{y})$, which takes in vectors $\vec{x} \in \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^k$ and outputs a scalar $\in \mathbb{R}$. That is, $f(\vec{x},\vec{y})$ is $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$. With this new model, how can we adapt our previous linearization method?

One way to linearize the function f is to do it for every single element in $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top$ and $\vec{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_k \end{bmatrix}^\top$. Then, when we are looking at x_i or y_j, we fix everything else as constant. This would give us the linear approximation

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_\star,\vec{y}_\star) + \sum_{i=1}^{n} \frac{\partial f(\vec{x},\vec{y})}{\partial x_i}(x_i - x_{\star,i}) + \sum_{j=1}^{k} \frac{\partial f(\vec{x},\vec{y})}{\partial y_j}(y_j - y_{\star,j}). \tag{19}$$

In order to simplify this equation, we can define the rows $D_{\vec{x}}$ and $D_{\vec{y}}$ as

$$D_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}, \tag{20}$$

$$D_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \dots & \frac{\partial f}{\partial y_k} \end{bmatrix}. \tag{21}$$

Then, Equation (19) can be rewritten as

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_\star,\vec{y}_\star) + (D_{\vec{x}} f)\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{x} - \vec{x}_\star) + (D_{\vec{y}} f)\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{y} - \vec{y}_\star). \tag{22}$$

Assume that $n = k$ and we define the function $f(\vec{x},\vec{y}) = \vec{x}^\top \vec{y} = \sum_{i=1}^{k} x_i y_i$. Find $D_{\vec{x}} f$ and $D_{\vec{y}} f$.

Answer: The derivative is a *row* vector (as denoted above), so if we apply the definition (and write out the given function explicitly as $x_1 y_1 + x_2 y_2 + \ldots + x_k y_k$), we have:

$$D_{\vec{x}} f = \vec{y}^\top \tag{23}$$

and

$$D_{\vec{y}} f = \vec{x}^\top. \tag{24}$$

(f) Following the above part, find the linear approximation of $f(\vec{x},\vec{y})$ near $\vec{x}_\star = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{y}_\star = \begin{bmatrix}-1\\2\end{bmatrix}$.

Recall that $f(\vec{x},\vec{y}) = \vec{x}^\top\vec{y} = \sum_{i=1}^{k} x_i y_i$.

Answer: From the solution in the previous part, we can write

$$f(\vec{x},\vec{y}) \approx f(\vec{x}_\star,\vec{y}_\star) + (D_{\vec{x}}f)\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{x}-\vec{x}_\star) + (D_{\vec{y}}f)\Big|_{(\vec{x}_\star,\vec{y}_\star)} \cdot (\vec{y}-\vec{y}_\star) \tag{25}$$

$$= \vec{x}_\star^\top\vec{y}_\star + \vec{y}_\star^\top(\vec{x}-\vec{x}_\star) + \vec{x}_\star^\top(\vec{y}-\vec{y}_\star). \tag{26}$$

Putting in $\vec{x}_\star = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{y}_\star = \begin{bmatrix}-1\\2\end{bmatrix}$, and let's find the approximation of $f\left(\begin{bmatrix}1+\delta_1\\2+\delta_2\end{bmatrix}, \begin{bmatrix}-1+\delta_3\\2+\delta_4\end{bmatrix}\right)$, we have

$$f\left(\begin{bmatrix}1+\delta_1\\2+\delta_2\end{bmatrix}, \begin{bmatrix}-1+\delta_3\\2+\delta_4\end{bmatrix}\right) \approx \vec{x}_\star^\top\vec{y}_\star + \vec{y}_\star^\top(\vec{x}-\vec{x}_\star) + \vec{x}_\star^\top(\vec{y}-\vec{y}_\star) \tag{27}$$

$$= 3 + \begin{bmatrix}-1 & 2\end{bmatrix}\begin{bmatrix}\delta_1\\\delta_2\end{bmatrix} + \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}\delta_3\\\delta_4\end{bmatrix} \tag{28}$$

$$= 3 - \delta_1 + 2\delta_2 + \delta_3 + 2\delta_4. \tag{29}$$

Let's compare this with the true value $f\left(\begin{bmatrix}1+\delta_1\\2+\delta_2\end{bmatrix}, \begin{bmatrix}-1+\delta_3\\2+\delta_4\end{bmatrix}\right)$ We have:

$$f\left(\begin{bmatrix}1+\delta_1\\2+\delta_2\end{bmatrix}, \begin{bmatrix}-1+\delta_3\\2+\delta_4\end{bmatrix}\right) = (1+\delta_1)(-1+\delta_3) + (2+\delta_2)(2+\delta_4) \tag{30}$$

$$= 3 - \delta_1 + 2\delta_2 + \delta_3 + 2\delta_4 + \delta_1\delta_3 + \delta_2\delta_4. \tag{31}$$

As we can see, our approximation removes the second order δ terms $\delta_1\delta_3$ and $\delta_2\delta_4$, which is valid for small δ_i.

(g) When the function $\vec{f}(\vec{x}, \vec{y}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$ takes in vectors and outputs a vector (rather than a scalar), we can view each dimension in $\vec{f}$ independently as a separate function f_i, and linearize each of them:

$$\vec{f}(\vec{x}, \vec{y}) = \begin{bmatrix} f_1(\vec{x}, \vec{y}) \\ f_2(\vec{x}, \vec{y}) \\ \vdots \\ f_m(\vec{x}, \vec{y}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}} f_1 \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}} f_1 \cdot (\vec{y} - \vec{y}_\star) \\ f_2(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}} f_2 \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}} f_2 \cdot (\vec{y} - \vec{y}_\star) \\ \vdots \\ f_m(\vec{x}_\star, \vec{y}_\star) + D_{\vec{x}} f_m \cdot (\vec{x} - \vec{x}_\star) + D_{\vec{y}} f_m \cdot (\vec{y} - \vec{y}_\star) \end{bmatrix} \tag{32}$$

We can rewrite this in a clean way with the *Jacobian*:

$$D_{\vec{x}} \vec{f} = \begin{bmatrix} D_{\vec{x}} f_1 \\ D_{\vec{x}} f_2 \\ \vdots \\ D_{\vec{x}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \tag{33}$$

and similarly

$$D_{\vec{y}} \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_k} \end{bmatrix}. \tag{34}$$

Then, the linearization becomes

$$\vec{f}(\vec{x}, \vec{y}) \approx \vec{f}(\vec{x}_\star, \vec{y}_\star) + (D_{\vec{x}} \vec{f})\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{x} - \vec{x}_\star) + (D_{\vec{y}} \vec{f})\Big|_{(\vec{x}_\star, \vec{y}_\star)} \cdot (\vec{y} - \vec{y}_\star). \tag{35}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$. Find $D_{\vec{x}} \vec{f}$, applying the definition above.

Answer: Here, we have

$$D_{\vec{x}} \vec{f} = \begin{bmatrix} 2x_1 x_2 & x_1^2 \\ x_2^2 & 2x_1 x_2 \end{bmatrix}. \tag{36}$$

(h) Compare the approximation of $\vec{f}$ at the point $\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}$ using $\vec{x}_\star = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ versus $\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right)$. Recall the definition that $\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 x_2 \\ x_1 x_2^2 \end{bmatrix}$.

Answer: Let $\delta = 0.01$. The true value is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) = \begin{bmatrix} (2+\delta)^2(3+\delta) \\ (2+\delta)(3+\delta)^2 \end{bmatrix} = \begin{bmatrix} 12 + 16\delta + 7\delta^2 + \delta^3 \\ 18 + 21\delta + 8\delta^2 + \delta^3 \end{bmatrix}. \tag{37}$$

On the other hand, our approximation is

$$\vec{f}\left(\begin{bmatrix} 2.01 \\ 3.01 \end{bmatrix}\right) \approx \vec{f}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + \begin{bmatrix} 12 & 4 \\ 9 & 12 \end{bmatrix} \cdot \begin{bmatrix} \delta \\ \delta \end{bmatrix} = \begin{bmatrix} 12 + 16\delta \\ 18 + 21\delta \end{bmatrix}. \tag{38}$$

Again, our approximation essentially removes the higher order terms of δ.
When we plug in $\delta = 0.01$, we have

$$\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right) = \begin{bmatrix}12.160701\\18.210801\end{bmatrix} \tag{39}$$

and our approximation is

$$\vec{f}\left(\begin{bmatrix}2.01\\3.01\end{bmatrix}\right) = \begin{bmatrix}12.16\\18.21\end{bmatrix}. \tag{40}$$

(i) **Practice:** Let $\vec{x}$ and $\vec{y}$ be vectors with 2 rows, and let $\vec{w}$ be another vector with 2 rows. Let $\vec{f}(\vec{x},\vec{y}) = \vec{x}\vec{y}^\top\vec{w}$. Find $D_{\vec{x}}\vec{f}$ and $D_{\vec{y}}\vec{f}$.

Answer: Here, recall that

$$\vec{f} = \begin{bmatrix}x_1\\x_2\end{bmatrix}\cdot\begin{bmatrix}y_1 & y_2\end{bmatrix}\cdot\begin{bmatrix}w_1\\w_2\end{bmatrix} = \begin{bmatrix}x_1y_1 & x_1y_2\\x_2y_1 & x_2y_2\end{bmatrix}\cdot\begin{bmatrix}w_1\\w_2\end{bmatrix} = \begin{bmatrix}x_1y_1w_1 + x_1y_2w_2\\x_2y_1w_1 + x_2y_2w_2\end{bmatrix}. \tag{41}$$

Then,

$$D_{\vec{x}}\vec{f} = \begin{bmatrix}\frac{\partial f_1}{x_1} & \frac{\partial f_1}{x_2}\\\frac{\partial f_2}{x_1} & \frac{\partial f_2}{x_2}\end{bmatrix} = \begin{bmatrix}y_1w_1 + y_2w_2 & 0\\0 & y_1w_1 + y_2w_2\end{bmatrix} \tag{42}$$

and

$$D_{\vec{y}}\vec{f} = \begin{bmatrix}\frac{\partial f_1}{y_1} & \frac{\partial f_1}{y_2}\\\frac{\partial f_2}{y_1} & \frac{\partial f_2}{y_2}\end{bmatrix} = \begin{bmatrix}x_1w_1 & x_1w_2\\x_2w_1 & x_2w_2\end{bmatrix}. \tag{43}$$

We can also write

$$D_{\vec{x}}\vec{f} = \vec{y}^\top\vec{w}\cdot I \tag{44}$$

and

$$D_{\vec{y}}\vec{f} = \vec{x}\vec{w}^\top, \tag{45}$$

which can be derived by noticing that $\vec{y}^\top\vec{w} = \vec{w}^\top\vec{y}$.

(j) **Practice:** Continuing the above part, find the linear approximation of $\vec{f}$ near $\vec{x} = \vec{y} = \begin{bmatrix}1\\1\end{bmatrix}$ and with $\vec{w} = \begin{bmatrix}2\\1\end{bmatrix}$.

Answer: We have

$$\begin{aligned}\vec{f}(\vec{x},\vec{y}) &\approx \vec{f}(\vec{x}_\star,\vec{y}_\star) + D_{\vec{x}}\vec{f}\cdot(\vec{x}-\vec{x}_\star) + D_{\vec{y}}\vec{f}\cdot(\vec{y}-\vec{y}_\star) & (46)\\ &= \begin{bmatrix}3\\3\end{bmatrix} + \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}x_1-1\\x_2-1\end{bmatrix} + \begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}\cdot\begin{bmatrix}y_1-1\\y_2-1\end{bmatrix} & (47)\end{aligned}$$

(48)

Let's do an approximation of $\vec{f}\left(\begin{bmatrix}1+\delta_1\\1+\delta_2\end{bmatrix},\begin{bmatrix}1+\delta_3\\1+\delta_4\end{bmatrix}\right)$, then,

$$\vec{f}\left(\begin{bmatrix}1+\delta_1\\1+\delta_2\end{bmatrix},\begin{bmatrix}1+\delta_3\\1+\delta_4\end{bmatrix}\right) \approx \begin{bmatrix}3\\3\end{bmatrix} + \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\cdot\begin{bmatrix}\delta_1\\\delta_2\end{bmatrix} + \begin{bmatrix}2 & 1\\2 & 1\end{bmatrix}\cdot\begin{bmatrix}\delta_3\\\delta_4\end{bmatrix} = \begin{bmatrix}3+3\delta_1+2\delta_3+\delta_4\\3+3\delta_2+2\delta_3+\delta_4\end{bmatrix}.$$

We can compare with the true value

$$\begin{aligned}\vec{f}\left(\begin{bmatrix}1+\delta_1\\1+\delta_2\end{bmatrix},\begin{bmatrix}1+\delta_3\\1+\delta_4\end{bmatrix}\right) &= \begin{bmatrix}1+\delta_1\\1+\delta_2\end{bmatrix}\begin{bmatrix}1+\delta_3 & 1+\delta_4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\\ &= \begin{bmatrix}1+\delta_1+\delta_3+\delta_1\delta_3 & 1+\delta_1+\delta_4+\delta_1\delta_4\\1+\delta_2+\delta_3+\delta_2\delta_3 & 1+\delta_2+\delta_4+\delta_2\delta_4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\\ &= \begin{bmatrix}3+3\delta_1+2\delta_3+\delta_4+2\delta_1\delta_3+\delta_1\delta_4\\3+3\delta_2+2\delta_3+\delta_4+2\delta_2\delta_3+\delta_2\delta_4\end{bmatrix},\end{aligned} \tag{49}$$

and we see that our approximation removes the second order δ terms $\delta_1\delta_3$, $\delta_1\delta_4$, $\delta_2\delta_3$ and $\delta_2\delta_4$.

These linearizations are important for us because we can do many easy computations using linear functions.

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