EECS 16B Spring 2021 | Designing Information Devices and Systems II Discussion Worksheet | **Discussion 11B**

1. Understanding the SVD

We can compute the SVD for a wide matrix A with dimension $m \times n$ where $n > m$ using $A^\top A$ with the method covered in lecture. However, when doing so, you may realize that $A^\top A$ is much larger than $AA^\top$ for such wide matrices. This makes it more efficient to find the eigenvalues for $AA^\top$. In this question, we will explore how to compute the SVD using $AA^\top$ instead of $A^\top A$.

(a) What are the dimensions of $AA^\top$ and $A^\top A$?

Answer: Since A is $m \times n$, $AA^\top$ is $(m \times n) \times (n \times m)$, which is $m \times m$. Similarly $A^\top A$ is $(n \times m) \times (m \times n)$ which is $n \times n$.

(b) Given that the SVD of A is $A = U\Sigma V^\top$, find a symbolic expression for $AA^\top$.

Answer:

$$AA^\top = U\Sigma \underbrace{V^\top V}_{I} \Sigma^\top U^\top \tag{1}$$

$$= U\Sigma\Sigma^\top U^\top \tag{2}$$

(c) Using the solution to the previous part, how can we find U and Σ from $AA^\top$?

Answer: Knowing that $AA^\top$ is a symmetric matrix, we know that its normalized eigenvectors will be orthonormal.

From the properties of the SVD, we know that U is an orthonormal matrix of dimension $m \times m$ and $\Sigma\Sigma^\top$ is an $m \times m$ diagonal matrix, with the entries on the diagonal being ${\sigma_i}^2$. Each σ_i is a *singular value* of A.

We can calculate U by diagonalizing the symmetric matrix $AA^\top$. By the spectral theorem for real symmetric matrices, we will get an orthonormal basis of eigenvectors. The square root of the corresponding eigenvalues of $AA^\top$ will give us the singular values σ_i.

We can then construct Σ by putting these on the diagonal of a zero matrix with the same dimensions as A, and the corresponding eigenvectors will form the U matrix.

(d) Now that we have found the singular values σ_i and the corresponding vectors $\vec{u}_i$ in the matrix U, can you find the corresponding vectors $\vec{v}_i$ in V?

Answer: We know everything except for V. In particular, we know $\vec{u}_i$ is an eigenvector of $AA^\top$ with eigenvalue σ_i^2. Then

$$AA^\top \vec{u}_i = \sigma_i^2 \vec{u}_i \tag{3}$$

$$A^\top AA^\top \vec{u}_i = A^\top(\sigma_i^2 \vec{u}_i) \tag{4}$$

$$A^\top A(A^\top \vec{u}_i) = \sigma_i^2 (A^\top \vec{u}_i). \tag{5}$$

So we see that $A^\top \vec{u}_i$ is an eigenvector of $A^\top A$ with eigenvalue σ_i^2. Define $\vec{v}_i = \frac{A^\top \vec{u}_i}{\|A^\top \vec{u}_i\|}$. Then

$$\vec{v}_i = \frac{A^\top \vec{u}_i}{\|A^\top \vec{u}_i\|} \tag{6}$$

$$= \frac{A^\top \vec{u}_i}{\sqrt{\|A^\top \vec{u}_i\|^2}} \tag{7}$$

$$= \frac{A^\top \vec{u}_i}{\sqrt{(A^\top \vec{u}_i)^\top (A^\top \vec{u}_i)}} \tag{8}$$

$$= \frac{A^\top \vec{u}_i}{\sqrt{\vec{u}_i^\top A A^\top \vec{u}_i}} \tag{9}$$

$$= \frac{A^\top \vec{u}_i}{\sqrt{\vec{u}_i^\top \sigma_i^2 \vec{u}_i}} \tag{10}$$

$$= \frac{A^\top \vec{u}_i}{\sqrt{\sigma_i^2 \|\vec{u}_i\|^2}} \tag{11}$$

$$= \frac{A^\top \vec{u}_i}{\sqrt{\sigma_i^2}} \tag{12}$$

$$= \frac{A^\top \vec{u}_i}{\sigma_i}. \tag{13}$$

(e) Now we have a way to find the vectors $\vec{v}_i$ in matrix V! Verify that these vectors are orthonormal.

Answer: To verify that $\vec{v}_i$ in V are orthonormal, we must show that:

i. $\vec{v}_i$ are mutually orthogonal

ii. each $\vec{v}_i$ has norm 1.

Orthogonality:

To show orthogonality, we must show that any two vectors $\vec{v}_i = \frac{A^\top \vec{u}_i}{\sigma_i}$ and $\vec{v}_j = \frac{A^\top \vec{u}_j}{\sigma_j}$, with $i \neq j$, have an inner product of zero. Writing the inner product out:

$$\vec{v}_i^\top \vec{v}_j = \frac{\vec{u}_i^\top A}{\sigma_i} \frac{A^\top \vec{u}_j}{\sigma_j} \tag{14}$$

$$= \frac{\vec{u}_i^\top A A^\top \vec{u}_j}{\sigma_i \sigma_j} \tag{15}$$

$$= \frac{(\sigma_j)^2 \vec{u}_i^\top \vec{u}_j}{\sigma_i \sigma_j} \tag{16}$$

$$= 0 \tag{17}$$

In going from eq. (15) to eq. (16), we could have substituted the matrix product $AA^\top$ with the answer of part c) and simplified. Here, we recognize that the inner matrix $\Sigma\Sigma^\top$ is diagonal with σ_i on the diagonals. This is because we know that $\vec{u}_i$ and $\vec{u}_j$ are orthonormal as they are eigenvectors of a symmetric matrix $AA^\top$.

Thus for all $i \neq j$,

$$\vec{v}_i^\top \vec{v}_j = 0 \tag{18}$$

Norm of 1: If we follow the steps above with $i = j$, then we see that:

$$\vec{v}_i^\top \vec{v}_j = \vec{v}_i^\top \vec{v}_i \tag{19}$$

$$= \frac{(\sigma_i)^2 \vec{u}_i^\top \vec{u}_i}{\sigma_i \sigma_i} \tag{20}$$

$$= \frac{(\sigma_i)^2}{(\sigma_i)^2} \vec{u}_i^\top \vec{u}_i \tag{21}$$

$$= 1 \tag{22}$$

(f) Now that we have found $\vec{v}_i$, you may notice that we only have $m < n$ vectors of dimension n. This is not enough for a basis. How would you complete the m vectors to form an orthonormal basis?

Answer: We would use Gram-Schmidt.

If we append the standard basis for n-dimensional space, and orthonormalize, this will give us the desired result. The augmented collection of $n + m$ vectors certainly spans the whole space, and so after orthonormalization, we will have a collection of orthonormal vectors that spans the whole space. Along the way, some vectors will be found to be linearly dependent on those that came before — this is fine, we'll discard these. At the end, we will have n orthonormal vectors, the first set of which are the original $\vec{v}_i$.

(g) (Practice.) Given that $A = U\Sigma V^\top$ verify that the vectors you found to extend the $\vec{v}_i$ into a basis are in the nullspace of A.

Answer: Let $V = \begin{bmatrix} V_s & R \end{bmatrix}$ where V_s are the $\{\vec{v}_i\}$ we found using the $\{\vec{u}_i\}$ and R is composed of the remaining vectors found using Gram Schmidt. Let S be an $m \times m$ diagonal square matrix with σ_i on the diagonal (σ_i is allowed to be zero) such that $\Sigma = \begin{bmatrix} S & 0 \end{bmatrix}$ where 0 denotes filling in the remaining matrix dimensions with zeros.

$$A = U\Sigma V^\top = U \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} V_s^\top \\ R^\top \end{bmatrix} \tag{23}$$

And so:

$$AR = U \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} V_s^\top \\ R^\top \end{bmatrix} R \tag{24}$$

$$= U \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} 0 \\ R^\top R \end{bmatrix} \tag{25}$$

$$= U \begin{bmatrix} 0 \end{bmatrix} = 0 \tag{26}$$

Thus, everything in the subspace spanned by R maps to $\vec{0}$, and this shows that the subspace is in the nullspace of A.

(h) Using the previous parts of this question and what you learned from lecture, write out a procedure on how to find the SVD for *any* matrix.

Answer: We calculate the SVD of matrix A as follows.

i. Pick $A^\top A$ or $AA^\top$ — whichever one is smaller.

ii. i. If using $A^\top A$, find the eigenvalues λ_i of $A^\top A$ and order them, so that $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $\lambda_{r+1} = \cdots = \lambda_n = 0$.

   If using $AA^\top$, find its eigenvalues $\lambda_1, \ldots, \lambda_m$ and order them the same way.

   ii. If using $A^\top A$, find orthonormal eigenvectors $\vec{v}_i$ such that

$$A^\top A \vec{v}_i = \lambda_i \vec{v}_i, \quad i = 1, \ldots, r$$

   If using $AA^\top$, find orthonormal eigenvectors $\vec{u}_i$ such that

$$AA^\top \vec{u}_i = \lambda_i \vec{u}_i, \quad i = 1, \ldots, r$$

   iii. Set $\sigma_i = \sqrt{\lambda_i}$.

   If using $A^\top A$, obtain $\vec{u}_i$ from $\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i, \quad i = 1, \ldots, r$.

   If using $AA^\top$, obtain $\vec{v}_i$ from $\vec{v}_i = \frac{1}{\sigma_i} A^\top \vec{u}_i, \quad i = 1, \ldots, r$.

iii. If you want to completely construct the U or V matrix, complete the basis (or columns of the appropriate matrix) using Gram-Schmidt to get a full orthonormal matrix.

The full matrix form of SVD is taken to better understand the matrix A in terms of the 3 nice matrices U, Σ, V. Often in practice, we do not completely construct the U and V matrices. After all, in many applications, we don't need all the vectors.

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