EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet Discussion 9A

This is a review session consisting of past 16B exam problems to help you review for the midterm exam tonight.

1. Transfer Function Analysis (Fa20 MT1, Q11)

This question was presented in the form of multiple-choice and True or False; it has been adapted to mimic a format closer to this year's exam.

You are given the following transfer function:

$$H(\omega) = \frac{1}{10} \frac{\left(1 + \frac{j\omega}{100}\right)(1 + j\omega \cdot 10)}{\left(1 + \frac{j\omega}{1000}\right)(1 + j\omega)} \tag{1}$$

(a) What is the magnitude of eq. (1), $|H(\omega)|$, as $\omega \to 0$?

Answer: Since the denominator of the transfer function is defined when $\omega = 0$, we can directly plug in this frequency to the find the value of the transfer function. We find that:

$$\begin{aligned}\lim_{\omega \to 0} H(\omega) &= \frac{1}{10}\left(\frac{1 \cdot 1}{1 \cdot 1}\right) \\ &= \frac{1}{10}\end{aligned}$$

(b) What is $|H(\omega)|$ as $\omega \to \infty$?

Answer: Here, we cannot directly plug in $\omega = \infty$ since both the numerator and denominator will go to infinity, and this is undefined/indeterminate. To avoid this issue, we can use the same approach to evaulate this limit as used in dis06A, and convert our expression so the ω terms only appear in denominators. Then, we can safely plug in $\omega = \infty$ and rather than blowing up, those fractions will go to 0. We can manipulate $H(\omega)$ by dividing top and bottom by $(j\omega)^2$. Note that *each* pole and zero term needs to be divided by $j\omega$, so we need 2 of these factors for the numerator and also 2 for the denominator.:

$$\begin{aligned}H(\omega) &= \frac{1}{10} \frac{\left(1 + \frac{j\omega}{100}\right)(1 + j\omega \cdot 10)}{\left(1 + \frac{j\omega}{1000}\right)(1 + j\omega)} \\ &= \frac{1}{10} \frac{\frac{\left(1+\frac{j\omega}{100}\right)}{j\omega} \frac{(1+j\omega \cdot 10)}{j\omega}}{\frac{\left(1+\frac{j\omega}{1000}\right)}{j\omega} \frac{(1+j\omega)}{j\omega}} \\ &= \frac{1}{10} \frac{\left(\frac{1}{j\omega} + \frac{1}{100}\right)\left(\frac{1}{j\omega} + 10\right)}{\left(\frac{1}{j\omega} + \frac{1}{1000}\right)\left(\frac{1}{j\omega} + 1\right)}\end{aligned}$$

Now, we can evaluate the limit as $\omega \to \infty$ directly! Doing so, we find:

$$\begin{aligned}\lim_{\omega\to\infty} H(\omega) &= \frac{1}{10}\frac{\left(0+\frac{1}{100}\right)(0+10)}{\left(0+\frac{1}{1000}\right)(0+1)} \\ &= \frac{1}{10}\frac{\frac{1}{100}\cdot 10}{\frac{1}{1000}\cdot 1} \\ &= \frac{1}{10}100 \\ &= 10\end{aligned}$$

(c) Now, suppose we are working with the modified transfer function below.

$$H_M(\omega) = 10\frac{(1+j\omega\cdot 10)}{(1+j\omega)} \tag{2}$$

Construct the Phase Bode Plot of this transfer function. Be sure to convert the transfer function to its rational form for easy inspection of the poles and zeros. Use it to estimate the phase of the transfer function, $\measuredangle H(\omega)$, at $\omega_1 = 1$ and $\omega_2 = 100$.

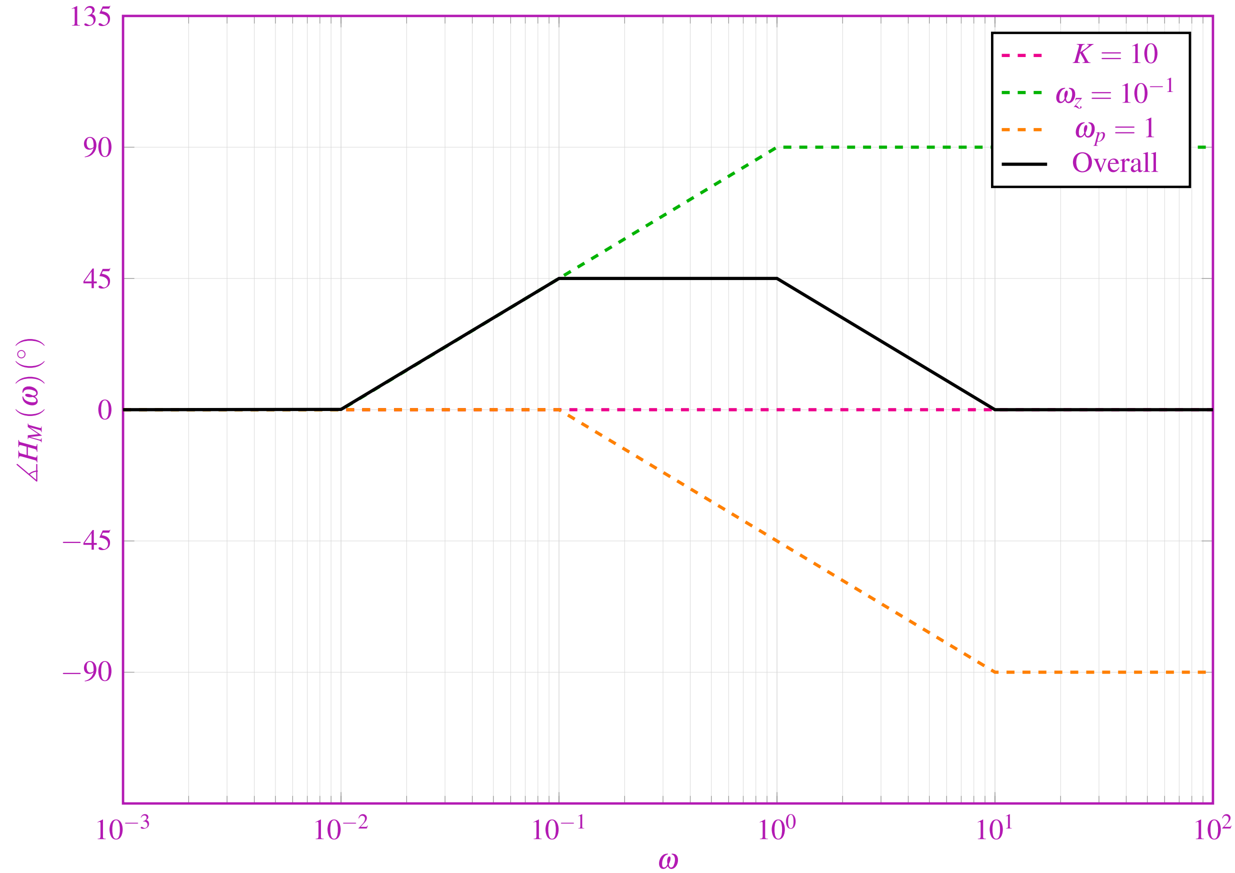
Answer: To compose the Bode Plot, we first convert $H_M(\omega)$ to the rational transfer function form. We're nearly there! But the numerator has a $1 + j\omega \cdot 10$ term, which isn't of the form $1+\frac{j\omega}{\omega_z}$. So, we convert it to $1+\frac{j\omega}{\frac{1}{10}}$. Our final transfer function, then, becomes:

$$H_{MR}(\omega) = 10\frac{\left(1+\frac{j\omega}{\frac{1}{10}}\right)}{(1+j\omega)} \tag{3}$$

Now, we see that the given transfer function has 3 key components that contribute to the phase in different ways; a gain term, a pole, and a zero. Note 6, Sec. 3.4 has a good refresher on the components; we describe them below.

- 10: gain term, contributes $0°$ phase at all frequencies.
- $1+j\omega\cdot 10 \equiv 1+\frac{j\omega}{\frac{1}{10}}$: This is a zero at a frequency of $\omega_z = \frac{1}{10}$.
  By the Bode plot approximation, a zero at some frequency $\omega_{c,z}$ has zero phase at frequencies over 10 times smaller than $\omega_{c,z}$. Then, the phase linearly *increases* to $+45°$ at exactly $\omega_{c,z}$, and continues to rise until $10\omega_{c,z}$. At $10\omega_{c,z}$, the phase will be $+90°$, which it stays at for all higher frequencies.
- $1+j\omega$: This is a pole at $\omega_p = 1$.
  By the Bode plot approximation, a pole at frequency $\omega_{c,p}$ has zero phase at frequencies over 10 times smaller than $\omega_{c,p}$. Then, the phase linearly *decreases* to $-45°$ at exactly $\omega_{c,p}$, and continues to decrease until $10\omega_{c,p}$. At $10\omega_{c,p}$, the phase will be $-90°$, which it stays at for all higher frequencies.

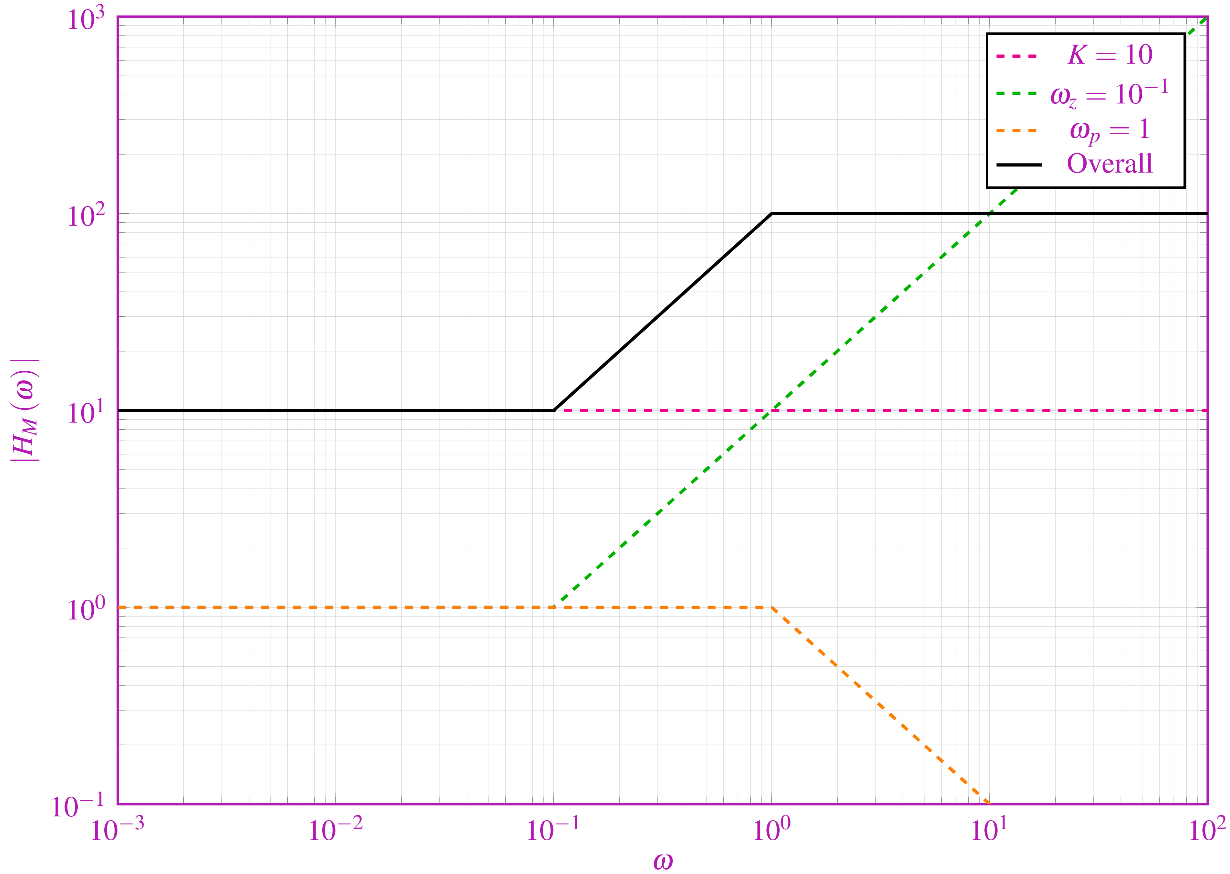
Looking at the plot below, we see how to plot each of these components individually, as well as how to add them together.



So the phase of the transfer function at $\omega_1 = 1$ is 45°. The phase at $\omega_2 = 100$ is 0° (both the pole and the zero happened at far lower frequencies. When added together, they effectively add zero total phase.).

(d) Now, compose the Bode Plot Approximation for the magnitude of the transfer function, $|H_M(\omega)|$. Use the rational form derived above.

Answer: We have: a gain term of 10, a zero at $\omega_z = \frac{1}{10}$, and a pole at $\omega_p = 1$. Below is a plot that composes and labels each of these elements, and gives the final answer.


$K = 10$
$\omega_z = 10^{-1}$
$\omega_p = 1$
Overall
$|H_M(\omega)|$
ω
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}

2. CMOS Circuits (Sp20, MT1, Q1)

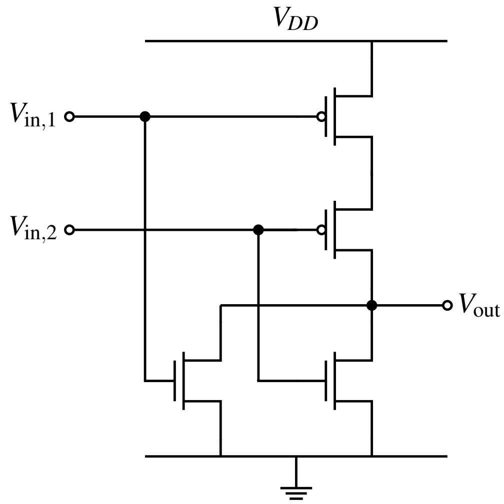


Figure 1: CMOS circuit

Consider the CMOS circuit of Figure 1. For each of the sets of $V_{\text{in},1}$ and $V_{\text{in},2}$ in the table below, fill in the corresponding voltage of the output V_{out}. You may assume that the threshold voltages for the transistors are $0 < V_{tn} < V_{DD}$ and $0 < |V_{tp}| < V_{DD}$.

$V_{\text{in},1}$	$V_{\text{in},2}$	V_o
0 V	0 V	
V_{DD}	0 V	
0 V	V_{DD}	
V_{DD}	V_{DD}	

Answer: We can first draw in the source/drain labels for each transistor.

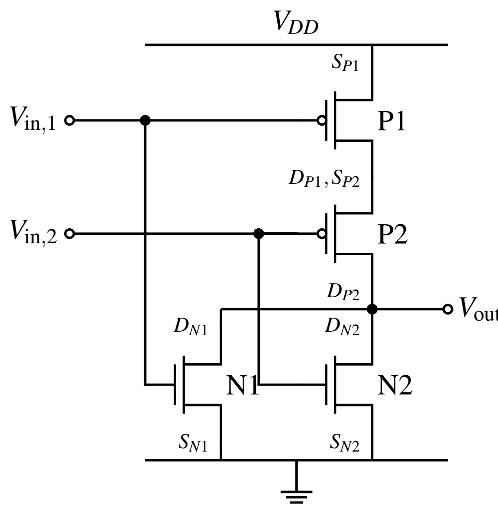


Figure 2: CMOS circuit

Now, we consider each case as follows:

- $V_{\text{in},1} = 0\,\text{V}$, $V_{\text{in},2} = 0\,\text{V}$: The NMOS devices have a low gate voltage, so they act as open switches since $V_{GS} < V_{tn}$. The PMOS devices are happy with a low gate voltage! Since $V_{SG} > |V_{tp}|$ (equivalently, $V_{GS} < -|V_{tp}|$), the PMOS switches are closed. Therefore, there is a path between V_{DD} and V_{out}.

- $V_{\text{in},1} = V_{DD}$, $V_{\text{in},2} = 0\,\text{V}$: The left NMOS device has a high gate voltage, so it acts as a closed switch ($V_{GS} > V_{tn}$). Oppositely, the right NMOS is an open switch since it's gate voltage is low. We only need one NMOS to form a path between ground and V_{out}, though, so we can say $V_{\text{out}} = 0$.

  For completeness, let's consider the PMOS devices. The top one is open since its gate voltage is high ($V_{SG} < |V_{tp}|$, $V_{GS} > -|V_{tp}|$). The bottom one is closed, but since the PMOS devices are in series, there's no complete path between V_{DD} and V_{out}.

- $V_{\text{in},1} = 0\,\text{V}$, $V_{\text{in},2} = V_{DD}$: This case is the opposite of that above; just swap left with right, and top with bottom. Again, $V_{\text{out}} = 0$.

- $V_{\text{in},1} = V_{DD}$, $V_{\text{in},2} = V_{DD}$: Here, both NMOS devices are closed ($V_{GS} > V_{tn}$). So, $V_{\text{out}} = 0$.
  Of the PMOS devices, both are open since their gate voltages are high ($V_{SG} < |V_{tp}|$, $V_{GS} > -|V_{tp}|$). Again, there's no complete path between V_{DD} and V_{out}.

$V_{\text{in},1}$	$V_{\text{in},2}$	V_{out}
$0\,\text{V}$	$0\,\text{V}$	V_{DD}
V_{DD}	$0\,\text{V}$	0
$0\,\text{V}$	V_{DD}	0
V_{DD}	V_{DD}	0

3. Simple Differential Equation to Review!

(a) What is the solution $x(t)$ to the following scalar differential equation?

$$\frac{d}{dt}x(t) = 3jx(t) \qquad x(0) = 4$$

Answer: We know that the exponential function gives the general form of the solution, and we can use the initial condition to populate the remaining constant. The general solution is:

$$x(t) = c_1 e^{3jt}$$

Using the initial condition, we find that $c_1 = x(0) = 4$. So the final solution is:

$$x(t) = 4e^{3jt}$$

Contributors:

- Neelesh Ramachandran.
- Mauricio Bustamante.
- Seth Sanders.