EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet Discussion 8B

The relevant note for this discussion is Note 9.

1. Eigenvalues Placement in Discrete Time

Consider the following linear discrete time system

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t] + \vec{w}[t] \tag{1}$$

(a) Is the system given in eq. (1) stable?

Answer: For notation's sake, let's write the system in the familiar form

$$\vec{x}[t+1] = A\vec{x}[t] + \vec{b}u[t] + \vec{w}[t]$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We have to calculate the eigenvalues of matrix A. Thus,

$$\det(\lambda I - A) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -2$$

Since the magnitudes of the eigenvalues λ_1 and λ_2 are greater than 1, the system is unstable.

(b) Derive a state space representation of the resulting closed loop system using state feedback of the form $u[t] = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[t]$.

Hint: If you're having trouble parsing this expression for $u[t]$, note that $\begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is a *row vector*, while $\vec{x}[t]$ is a *column vector*. What happens when we multiply a row vector with a column vector like this?

Answer: The closed loop system using state feedback has the form

$$\begin{aligned}
\vec{x}[t+1] &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t] \\
&= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[t] \right) \\
&= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \vec{x}(t) \\
&= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} \right) \vec{x}[t] \\
&= \underbrace{\begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}}_{A_{\text{cl}}} \vec{x}[t].
\end{aligned}$$

(c) Find the appropriate state feedback constants, k_1, k_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.

Answer: From the previous part we have computed the closed loop system as

$$\vec{x}[t+1] = \underbrace{\begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}}_{A_{\text{cl}}} \vec{x}[t]$$

Thus, finding the eigenvalues of the above system we have

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} k_1 - \lambda & 1+k_2 \\ 2 & -1-\lambda \end{bmatrix} = \lambda^2 + (1-k_1)\lambda + (-k_1 - 2k_2 - 2)$$

We want to place the eigenvalue at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. This means that we should choose the gains k_1 and k_2 so that the characteristic equation is

$$0 = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = \lambda^2 - \frac{1}{4}.$$

Thus we should choose k_1 and k_2 satisfying the system of equations

$$\begin{aligned} 0 &= 1 - k_1 \\ -\frac{1}{4} &= -k_1 - 2k_2 - 2 \end{aligned}.$$

This system has solution $k_1 = 1$, $k_2 = -\frac{11}{8}$.

(d) Is the system now stable?

Answer: Yes, since the closed loop system has eigenvalues with magnitude less than one, it is stable.

(e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t]$ as the way that the discrete-time control acted on the system. As before, we use $u[t] = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[t]$ to try and control the system. What would the eigenvalues be? Can you move all the eigenvalues to where you want? Give an intuitive explanation of what is going on.

Answer:

$$\vec{x}[t+1] = \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \vec{x}[t] \tag{2}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix} \right) \vec{x}[t] \tag{3}$$

(4)

Finding the eigenvalues λ:

$$\begin{aligned}
0 &= \det\left(\begin{bmatrix} k_1-\lambda & k_2+1 \\ k_1+2 & k_2-1-\lambda \end{bmatrix}\right) && (5)\\
&= (k_1-\lambda)(k_2-1-\lambda)-(k_1+2)(k_2+1) && (6)\\
&= k_1(k_2-1)-k_1\lambda-\lambda(k_2-1)+\lambda^2-(k_1k_2+k_1+2k_2+2) && (7)\\
&= k_1k_2-k_1-k_1\lambda-\lambda k_2+\lambda+\lambda^2-k_1k_2-k_1-2k_2-2 && (8)\\
&= \lambda^2+(1-k_1-k_2)\lambda-2(1+k_1+k_2) && (9)\\
&= (\lambda+2)(\lambda-(1+k_1+k_2)) && (10)
\end{aligned}$$

We can see that the eigenvalue at $\lambda=-2$ cannot be moved, so we cannot arbitrarily change our eigenvalues with this control input.

2. Controlling states by designing sequences of inputs

This is something that you saw in 16A in the Segway problem. In that problem, you were given a semi-realistic model for a segway. Here, we are just going to consider the following matrix chosen for ease of understanding what is going on:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \vec{b}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[t+1]=A\vec{x}[t]+\vec{b}u[t].$$

(a) We are given the initial condition $\vec{x}[0]=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}[T]=\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some specific $T\geq 0$. We don't need to stay there, we just want to be in this state at that time. What is the smallest T such that this is possible? What is our choice of sequence of inputs $u[t]$?

Answer: To ease notation, let

$$\vec{x}[t]=\begin{bmatrix} x_1[t] \\ x_2[t] \\ x_3[t] \\ x_4[t] \end{bmatrix}.$$

Writing out expressions for $x[t]$ we get:

$$\vec{x}[1]=A\vec{x}[0]+\vec{b}u[0]=\begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix},$$

$$\vec{x}[2] = \begin{bmatrix} x_3[0] \\ x_4[0] \\ u[0] \\ u[1] \end{bmatrix},$$

$$\vec{x}[3] = \begin{bmatrix} x_4[0] \\ u[0] \\ u[1] \\ u[2] \end{bmatrix},$$

and if $t \geq 4$,

$$\vec{x}[t] = \begin{bmatrix} u[t-4] \\ u[t-3] \\ u[t-2] \\ u[t-1] \end{bmatrix}.$$

Hence, the smallest T is equal to 4, with $u[0] = [1]$, $u[1] = [2]$, $u[2] = [3]$, $u[3] = [4]$.

(b) What if we started from $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest T and what is our choice of $u[t]$?

Answer: Looking over our expressions for x[t] from the previous part, we see that the earliest T whose expression can be set to the desired state is $T = 1$ requiring $u[0] = 4$.

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ u[0] \end{bmatrix}.$$

(c) What if we started from $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$? What is the smallest T and what is our choice of $u[t]$?

Answer: Looking over our expressions for x[t], we see that the earliest T whose expression can be set to the desired state in this case is $T = 4$ requiring $u[0] = [1]$, $u[1] = [2]$, $u[2] = [3]$, $u[3] = [4]$.

3. Uncontrollability

Consider the following discrete-time system with the given initial state:

$$\vec{x}[t+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[t]$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(a) Is the system controllable?

Answer:

$$\mathscr{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Since the controllability matrix $\mathscr{C}$ only has rank 2, the system is not controllable.

(b) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T-1$?

Answer: No, if we write out our expressions for $x[t]$ we can see:

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}$$

$$\vec{x}[2] = \begin{bmatrix} 4 \\ 2u[0]-6 \\ 2u[1]-3 \end{bmatrix}$$

$$\vec{x}[t] = \begin{bmatrix} 2^t x_0[0] \\ -3x_0[t-1]+x_2[t-1] \\ x_1[t-1]+2u[t-1] \end{bmatrix}$$

Note that in this expression for $x[t]$, $x_0[t] = 2^{t+1}$ is decoupled from all other states and inputs. From this expression we also see that there is no choice of inputs us to get to $x_0[T] = -2$. Therefore, we will never be able to reach $\vec{x}[T] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for any T.

(c) Is it possible to reach $\vec{x}[T] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some $t = T$? For what input sequence $u[t]$ up to $t = T-1$?

Answer: If we look at our expressions for $x[t]$ starting from $t = 1$, we see that we can set our inputs to reach the desired state in a single timestep ($T = 1$) by setting $u[0] = -2$.

$$\vec{x}[1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[0] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

Thus we see that a system being uncontrollable does not mean we are unable to reach anything at all, but just that the range that can be reached is limited.

(d) Find the set of all possible states reachable after two timesteps.

Answer:

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}$$

$$\vec{x}[2] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[1] = \begin{bmatrix} 4 \\ -6+2u[0] \\ -3+2u[1] \end{bmatrix}$$

Since we can set $u[0]$ and $u[1]$ arbitrarily, we can reach any state of the form $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$ after two timesteps.

Contributors:

- Ioannis Konstantakopoulos.
- John Maidens.
- Anant Sahai.
- Regina Eckert.
- Druv Pai.
- Kuan-Yun Lee.
- Kareem Ahmad.
- Titan Yuan.