EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet Discussion 8A

For this discussion, Note 8 is helpful.

1. Proctoring Practice

To prepare for the midterm coming up on **March 15, 2021**, let's take a minute to ensure that the proctoring system will work.

Find the email we sent you with a Zoom link (it should have subject line "[EECS 16B] Personal Zoom Proctoring Link for Exams"), join this meeting, and record yourself for five minutes.

In case something went wrong with your Zoom room, fill out this form to tell us what went wrong so we can fix it for you.

2. BIBO Stability

Consider a continuous-time scalar real differential equation with known solution

$$\frac{d}{dt}x(t) = ax(t) + bu(t) \qquad x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau.$$

Show that if the system has $\mathrm{Re}\{a\} > 0$, then a bounded input can result in an unbounded output (i.e., the system is unstable) for every initial condition $x(0)$.

Answer: To start, let's consider the case when $x(0) = 0$. Now, we are left with the integral term to show that a bounded input can result in an unbounded output. A bounded input u implies

$$|u(t)| \leq k \text{ for all } t.$$

Let's consider the case when $u(t) = k$ for all t, giving us

$$x(t) = \int_0^t e^{a(t-\tau)}bk\,d\tau.$$

Rearranging the integral, we can focus on the exponential term:

$$x(t) = bk\int_0^t e^{a(t-\tau)}\,d\tau.$$

For $a \neq 0$, with a change of variables, we can evaluate this integral

$$\int_0^t e^{a(t-\tau)}d\tau = \frac{e^{at}-1}{a}.$$

So, when $t \to \infty$, this will be unbounded since $\left|e^{at}\right| = e^{\mathrm{Re}\{a\}t}$ will grow exponentially for $\mathrm{Re}\{a\} > 0$. Thus the magnitude of

$$x(t) = bk\frac{e^{at}-1}{a}$$

grows without bound, so the system is unstable.

The case where the initial condition is nonzero, i.e., $x(0) \neq 0$, is handled similarly. Let $u(t)$ have magnitude k and sign equal to the sign of $x(0)$. Then, if both of these signs are positive,

$$x(t) = e^{at}x(0) + bk\frac{e^{at}-1}{a}$$

and if both of these are negative,

$$x(t) = e^{at}x(0) - bk\frac{e^{at}-1}{a} = -e^{at}\left|x(0)\right| - bk\frac{e^{at}-1}{a} = -\left(e^{at}\left|x(0)\right| + bk\frac{e^{at}-1}{a}\right)$$

and the magnitude of this goes to ∞ for the same reason.

3. Changing behavior through feedback

In this question, we discuss how feedback control can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[t+1] = 0.9x[t] + u[t] + w[t] \tag{1}$$

where $u[t]$ is the control input we get to apply based on the current state and $w[t]$ is the external disturbance, each at time t.

Is the system stable? If $|w[t]| \leq \varepsilon$, what can you say about $|x[t]|$ at all times t if you further assume that $u[t] = 0$ and the initial condition $x[0] = 0$? How big can $|x[t]|$ get?

Answer: The system is stable, as $\lambda = 0.9 \to |\lambda| < 1$. We can say that $|x[t]|$ is bounded at all time if the disturbance is bounded. The system is bounded-input, bounded output (BIBO) stable.

Unrolling the system's recursion and extrapolating the general form,

$$\begin{aligned}
x[0] &= 0 \\
x[1] &= w[0] \\
x[2] &= 0.9w[0] + w[1] \\
x[3] &= 0.9^2 w[0] + 0.9w[1] + w[2] \\
&\vdots \\
x[t] &= \sum_{k=0}^{t-1} 0.9^{t-k-1} w[k].
\end{aligned}$$

We can check that this form works by plugging it into our recursion:

$$x[t+1] = 0.9x[t] + w[t] = 0.9\left(\sum_{k=0}^{t-1} 0.9^{t-k-1} w[k]\right) + w[t] = \sum_{k=0}^{t-1} 0.9^{t-k} w[k] + w[t] = \sum_{k=0}^{t} 0.9^{t-k} w[k]$$

which is exactly what our formula predicts.

Thus

$$|x[t]| = \left|\sum_{k=0}^{t-1} 0.9^{t-k-1} w[k]\right| \leq \sum_{k=0}^{t-1} \left|0.9^{t-k-1} w[k]\right| = \sum_{k=0}^{t-1} 0.9^{t-k-1} \varepsilon.$$

In the limit as $t \to \infty$, by the geometric series formula,

$$|x[t]| \leq \frac{\varepsilon}{1-0.9} = 10\varepsilon$$

(b) Suppose that we decide to choose a control law $u[t] = kx[t]$ to apply in feedback. For what values of λ can you get the system to behave like:

$$x[t+1] = \lambda x[t] + w[t]? \tag{2}$$

How would you pick k?

(*Note*: In this case, $w[t]$ can be thought of like another input to the system, except we can't control it.)

(*Note*: In lecture we call this term f – for feedback – instead of k, but we use k here since it's a more traditional notation for feedback, and also lowercase f is confused with functions.)

Answer: We can control the system to have any value of λ, as long as we're not limited on the values of k.

$$x[t+1] = 0.9x[t] + kx[t] + w[t] = \lambda x[t] + w[t].$$

Fitting terms,

$$k = \lambda - 0.9.$$

Note we can get a $\lambda > 1$ if we so desire; there is nothing stopping us from putting arbitrarily big λ by the choice of k.

(c) For the previous part, which k would you choose to minimize how big $|x[t]|$ can get?

Answer: From eq. (2), in order to have the minimum bound on $|x[t]|$, $\lambda = 0$. To get this λ,

$$k = -0.9.$$

In the limit as $t \to \infty$ in this case,

$$|x[t]| \leq \frac{\varepsilon}{1-0} = \varepsilon$$

The minimum bound on $|x(t)| = \varepsilon$ is the same bound as on the disturbance.

(d) What if instead of a 0.9, we had a 3 in the original eq. (1). What, if anything, would change?

Answer: If our system were now,

$$x[t+1] = 3x[t] + u[t] + w[t], \tag{3}$$

the system would no longer be stable. However, we can still choose any eigenvalue λ using closed loop feedback. In this case,

$$k = \lambda - 3.$$

(e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t] + \vec{w}[t] \tag{4}$$

where we further assume that B is an invertible square matrix.
Suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[t] = F\vec{x}[t]$. For what values of matrix G can you get the system to behave like:

$$\vec{x}[t+1] = G\vec{x}[t] + \vec{w}[t]? \tag{5}$$

How would you pick F given knowledge of A, B and the desired goal dynamics G?

Answer: Since in this case our input is the same rank as our output, we can arbitrarily choose the matrix G. As long as B is invertible (as given), we can define:

$$\begin{aligned}
\vec{x}[t+1] &= A\vec{x}[t] + B\vec{u}[t] + \vec{w}[t] && (6)\\
&= A\vec{x}[t] + BF\vec{x}[t] + \vec{w}[t] && (7)\\
&= (A+BF)\vec{x}[t] + \vec{w}[t] && (8)\\
&= G\vec{x}[t] + \vec{w}[t] && (9)
\end{aligned}$$

Therefore, matching terms,

$$A + BF = G \implies F = B^{-1}(G - A).$$

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