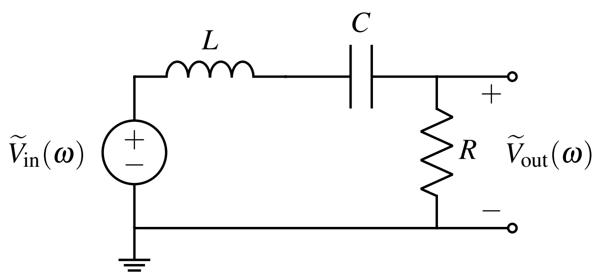
EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet Discussion 6B

For this discussion, Note 5 and Note 6 are helpful.

1. Band-pass filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



(a) **Write down the impedance of the series RLC combination in the form $Z_{RLC}(\omega) = A(\omega) + jX(\omega)$, where $A(\omega)$ and $X(\omega)$ are real valued functions of ω.**

Answer: Since the capacitor, resistor and inductor are in series, the equivalent impedance is given by

$$\begin{aligned} Z_{RLC}(\omega) &= Z_R(\omega) + Z_L(\omega) + Z_C(\omega) \\ &= R + j\omega L + \frac{1}{j\omega C} \\ &= R + j\left(\omega L - \frac{1}{\omega C}\right) \end{aligned}$$

so by pattern matching,

$$A(\omega) = R$$

and

$$X(\omega) = \omega L - \frac{1}{\omega C}$$

(b) **Write down the transfer function $H(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$ for this circuit.**

Answer: Using the same voltage divider rule we've used in the past, V_{out} is:

$$\begin{aligned} \widetilde{V}_{\text{out}} &= \widetilde{V}_{\text{in}} \frac{Z_R}{Z_{RLC}} \\ H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} &= \frac{R}{Z_{RLC}} \\ &= \frac{R}{R + j(\omega L - \frac{1}{\omega C})} \end{aligned}$$

(c) **At what frequency ω_n does $X(\omega_n) = 0$?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other. This is called the *natural frequency*.)

What happens to the relative magnitude of the impedances of the capacitor and inductor as ω moves above and below ω_n? What is the value of the transfer function at this frequency ω_n?

Answer:

$$X(\omega_n) = 0 = \omega_n L - \frac{1}{\omega_n C}.$$

Multiplying both sides by ω_n:

$$0 = \omega_n^2 L - \frac{1}{C}$$
$$\omega_n = \frac{1}{\sqrt{LC}}.$$

As the frequency ω increases above ω_n, the impedance of the inductor ($j\omega L$, which is directly proportional to ω) increases in magnitude, while the impedance of the capacitor ($1/(j\omega C)$, inversely proportional to ω) decreases in magnitude. Since the two components are in series, the impedance of the inductor will dominate, so $X(\omega) = \omega L - \frac{1}{\omega C} \approx \omega L$.

For the same reason, as the frequency ω decreases below ω_n, the impedance of the inductor decreases in magnitude, while the impedance of the capacitor decreases in magnitude. Thus the impedance of the capacitor will dominate, so $X(\omega) = \omega L - \frac{1}{\omega C} \approx -\frac{1}{\omega C}$,

At ω_n, $Z_{RLC} = R$, since the imaginary components cancel out perfectly. As a result

$$H(\omega_n) = \frac{R}{R} = 1.$$

(d) In most filters, we are interested in the cutoff frequency, since that helps define the frequency range over which the filter operates. Remember that this is the frequency at which the magnitude of the transfer function drops by a factor of $\sqrt{2}$ from its maximum value. Recalling that $H(\omega)$ can be written in the form $H(\omega) = \frac{R}{R+jX(\omega)}$, so that $\left|H(\omega)\right| = \frac{R}{\sqrt{R^2+X(\omega)^2}}$, we see that $\left|H(\omega)\right| \leq 1$. Thus, the cutoff frequency is ω_c such that $\left|H(\omega_c)\right| = \frac{1}{\sqrt{2}}$, which is when $X(\omega_c) = \pm R$.

We want to find such ω_c. To do this, we want to see what happens in the neighborhood of ω_n and so express the combined effect of the capacitor and inductor in terms of $\omega = \omega_n + \Delta\omega$, where $\Delta\omega$ is (presumably) a relatively small number compared to ω_n.

Write an expression for $X(\omega_n + \Delta\omega)$, where $\Delta\omega$ is a variable shift from ω_n. Find the values of $\Delta\omega_1$ and $\Delta\omega_2$ which give $X(\omega_n + \Delta\omega_1) = -R$, and $X(\omega_n + \Delta\omega_2) = +R$. You may use the approximation that $\frac{1}{1+x} \approx 1 - x$ if $x \ll 1$.

Answer: From the first part we know that,

$$X(\omega) = \omega L - \frac{1}{\omega C}.$$

So we get,

$$X(\omega_n + \Delta\omega) = (\omega_n + \Delta\omega)L - \frac{1}{(\omega_n + \Delta\omega)C}$$
$$= \omega_n L + \Delta\omega L - \frac{1}{\omega_n C(1 + \frac{\Delta\omega}{\omega_n})}$$

Using $\frac{1}{1+x} \approx 1 - x$ for $x \ll 1$ we get,

$$X(\omega_n + \Delta\omega) \approx \omega_n L + \Delta\omega L - \frac{1}{\omega_n C}\left(1 - \frac{\Delta\omega}{\omega_n}\right)$$

At the resonance condition, $\omega_n L = \frac{1}{\omega_n C}$. As a result we get,

$$X(\omega_n + \Delta\omega) \approx \Delta\omega L + \frac{\Delta\omega}{\omega_n^2 C}$$

From part (c) $\frac{1}{\omega_n^2} = LC$. We get,

$$X(\omega_n + \Delta\omega) \approx 2\Delta\omega L$$

Hence, if $X(\omega_n + \Delta\omega_2) = R$, then we get,

$$\Delta\omega_2 = \frac{R}{2L}$$

Similarly, when $X(\omega_n + \Delta\omega_1) = -R$, we get,

$$\Delta\omega_1 = \frac{-R}{2L}$$

Thus the cutoff frequencies are

$$\omega_c \approx \omega_n \pm \frac{R}{2L} = \frac{1}{\sqrt{LC}} \pm \frac{R}{2L}.$$

(e) **Simplify $X(\omega)$ in two cases, when $\omega \to \infty$ and when $\omega \to 0$. Plug this simplified $X(\omega)$ into your previously solved expressions to find the transfer function at high and low frequencies.**

Answer: At low frequencies,

$$X(\omega) = \omega L - \frac{1}{\omega C} \approx -\frac{1}{\omega C}$$

Plugging this back in to the original transfer function we get a CR high-pass filter.

$$H(\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \approx \frac{R}{R - j\frac{1}{\omega C}} = \frac{1}{1 - j\frac{1}{\omega RC}}.$$

We show that this is a high pass filter by verifying $\lim_{\omega\to\infty} \frac{1}{1-j\frac{1}{\omega RC}} = 1$ and $\lim_{\omega\to 0} \frac{1}{1-j\frac{1}{\omega RC}} = 0$.

At high frequencies,

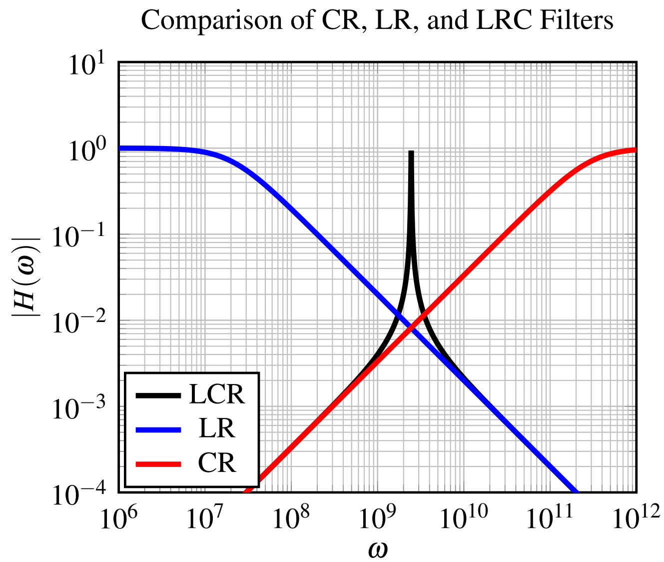
$$X(\omega) = \omega L - \frac{1}{\omega C} \approx \omega L$$

Plugging this back in to the original transfer function we get a LR low-pass filter.

$$H(\omega) = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \approx \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega\frac{L}{R}}$$

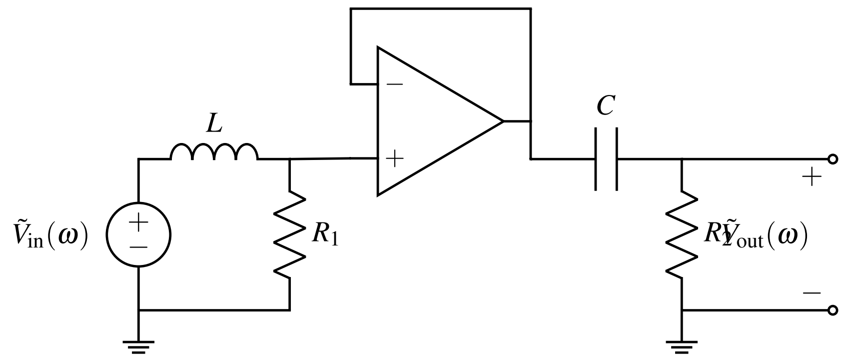
We show that this is a low pass filter by verifying $\lim_{\omega\to\infty} \frac{1}{1+j\omega\frac{L}{R}} = 0$ and $\lim_{\omega\to 0} \frac{1}{1+j\omega\frac{L}{R}} = 1$.

Our band-pass filter looks like an LR low-pass filter at high frequencies and a CR high-pass filter at low frequencies. Note that in this case, the cutoff frequencies for the LR and CR filters are not the same as LCR cutoff frequencies or the resonance frequency. In the vicinity of the resonance frequency, notice that the filter is much sharper than any first-order LR or RC filter.



2. Bode Plots

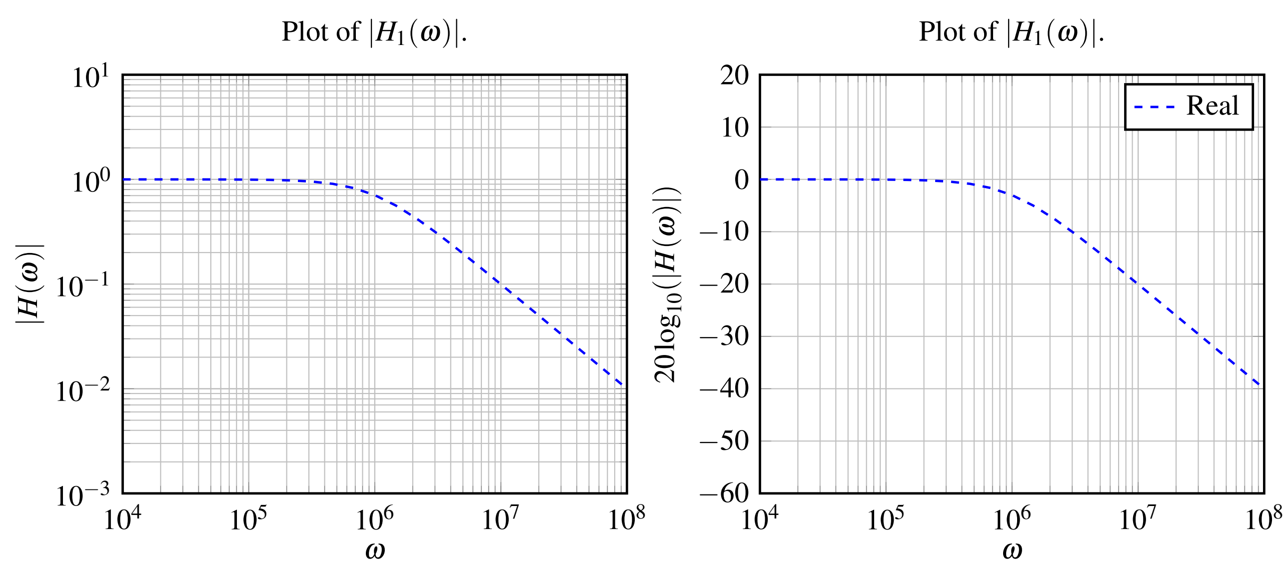
Our eventual goal is to construct Bode plots of the following circuit, with $L = 100\,\mu\text{H}$, $C = 1\,\mu\text{F}$, $R_1 = 100\,\Omega$, and $R_2 = 1\,\text{k}\Omega$.:



To do this we will leverage the fact that Bode plots can be composed in simple ways.

Before we start diving into the problem, let's consider a modification of the *magnitude* plot that will help us work with multiple magnitude plots at once. Namely, instead of plotting $|H(\omega)|$ vs. ω where the y-axis is on a *logarithmic* scale, we plot $20\log_{10}(|H(\omega)|)$ vs. ω instead, and now the y-axis is on a *linear* scale.

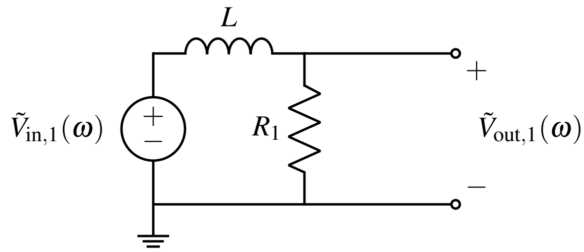
The reason that we do this is that, when combining transfer functions, we end up multiplying them. But we really want to add two plots graphically, not multiply them, so we will just plot and add the logarithms. (The constant multiple 20 is there for convention reasons.) Here's what this looks like, with the old grid on the left, and the new grid on the right:



Notice that we do not need to do this for the *phase* plots, since their y axes are naturally in linear scale, and combining plots can already be done by addition.

Now we are ready to begin working on the problems.

(a) Consider the first half of this circuit:



We learned in the previous discussion that the transfer function is given by

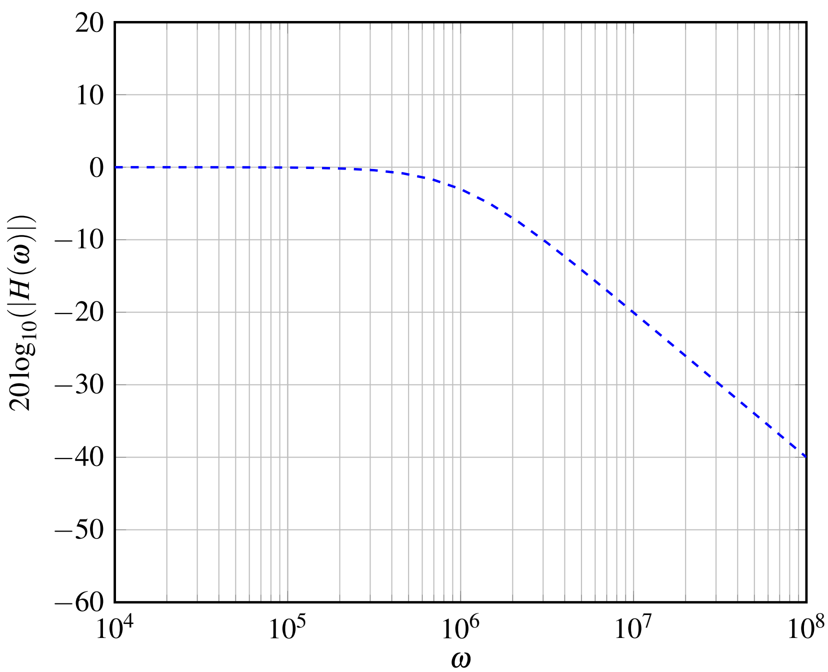
$$H_1(\omega) = \frac{\widetilde{V}_{\text{out},1}}{\widetilde{V}_{\text{in},1}} = \frac{1}{1 + j\omega\frac{L}{R_1}},$$

the cutoff frequency $\omega_{c,1}$ is given by

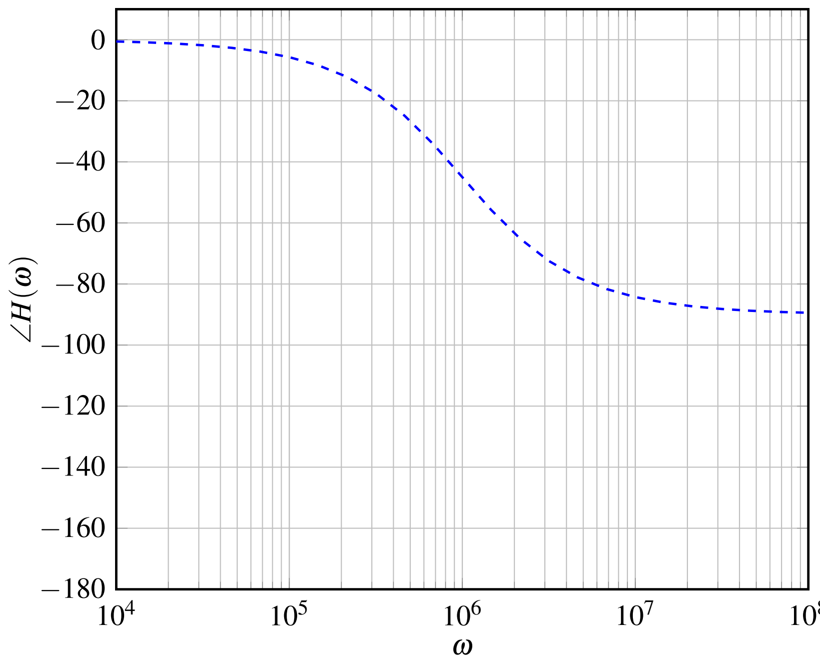
$$\omega_{c,1} = \frac{R_1}{L} = \frac{100\,\Omega}{100\,\mu\text{H}} = 1 \times 10^6\,\frac{\text{rad}}{\text{s}},$$

and plots of the transfer function are given by

Plot of $|H_1(\omega)|$.



Plot of $\angle H_1(\omega)$.



On these grids, **draw the Bode plots for magnitude and phase**.

Answer: We recognize that we can write $H_1(\omega)$ in the form

$$H_1(\omega) = \frac{1}{1 + j\omega\frac{L}{R_1}} = \frac{1}{1 + j\frac{\omega}{\omega_{c,1}}}.$$

Now we know the "recipe" to draw Bode plots, in particular

- For $\omega \ll \omega_{c,1}$,

$$H_1(\omega) = \frac{1}{1 + j\frac{\omega}{\omega_{c,1}}} \approx \frac{1}{1} = 1.$$

  What this means is that

  - For the Bode plot of $|H_1(\omega)|$ vs. ω:

  $$20\log_{10}(|H_1(\omega)|) \approx 20\log_{10}(1) = 0.$$

    Correspondingly, in the Bode plot, for $\omega < \omega_{c.1}$, the plot is constant with $20\log_{10}(|H_1(\omega)|) = 0$.

  - For the Bode plot of $\angle H_1(\omega)$ vs. ω:

  $$\angle H_1(\omega) \approx \angle 1 = 0.$$

    Correspondingly, in the Bode plot, for $\omega < \omega_{c.1}/10$, the plot is constant with $\angle H_1(\omega) = 0$.

- For $\omega \gg \omega_{c,1}$,

$$H_1(\omega) = \frac{1}{1 + j\frac{\omega}{\omega_{c,1}}} \approx \frac{1}{j\frac{\omega}{\omega_{c,1}}} = -j\frac{\omega_{c,1}}{\omega}.$$

  What this means is that

- For the Bode plot of $\left|H_1(\omega)\right|$ vs. ω:

$$20\log_{10}(\left|H_1(\omega)\right|) \approx 20\log_{10}\left(\frac{\omega_{c,1}}{\omega}\right) = 20\log_{10}(\omega_{c,1}) - 20\log_{10}(\omega).$$

  Correspondingly, in the Bode plot, for $\omega > \omega_{c.1}$, the plot, starting at $(\omega_{c,1}, 0)$, decreases with slope -20 per decade.

- For the Bode plot of $\angle H_1(\omega)$ vs. ω:

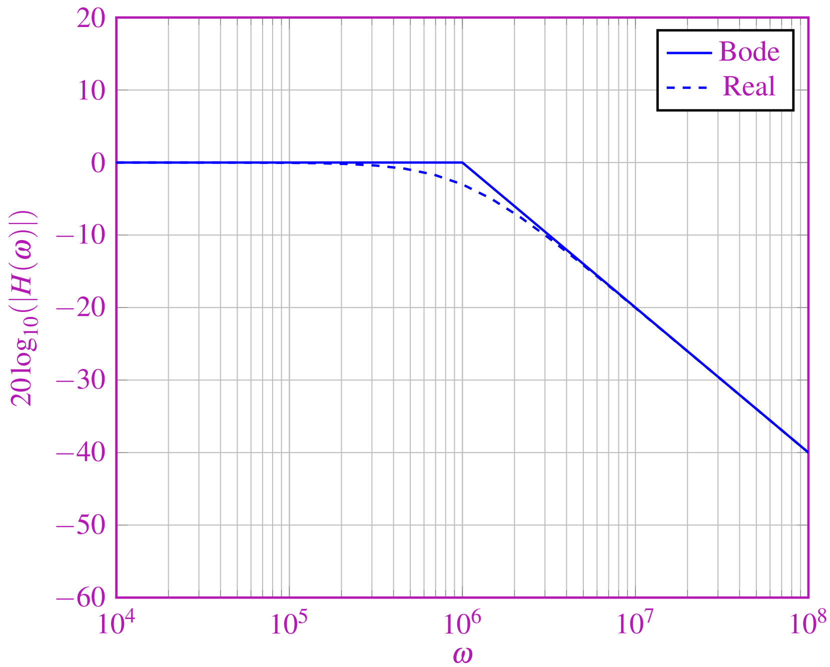
$$\angle H_1(\omega) \approx \angle\left(-j\frac{\omega_{c,1}}{\omega}\right) = \angle(-j) = -\frac{\pi}{2}.$$

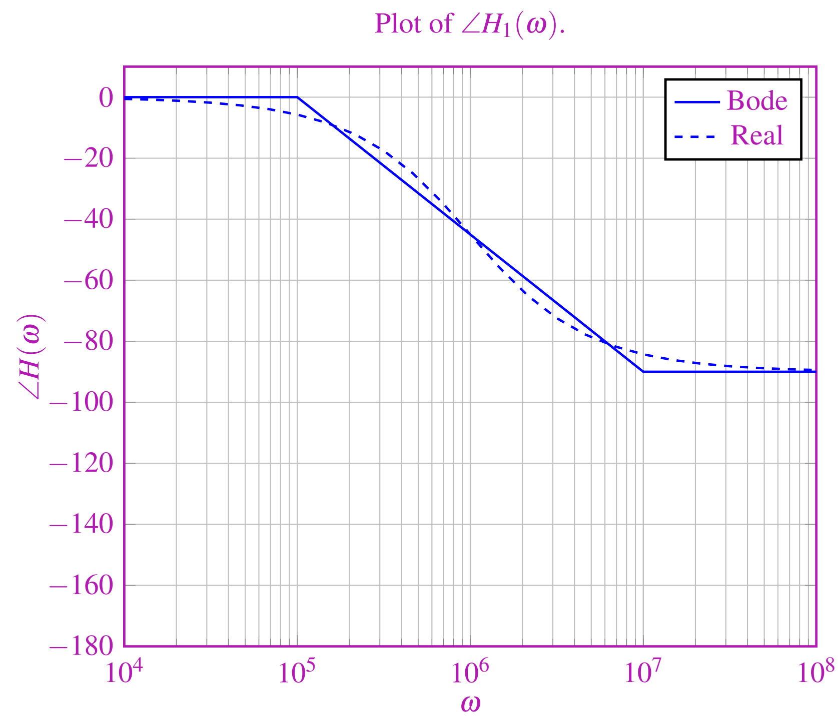
  Correspondingly, in the Bode plot, for $\omega > 10\omega_{c.1}$, the plot is constant with $\angle H_1(\omega) = -\frac{\pi}{2}$.

- For ω such that $\omega_{c,1}/10 < \omega < 10\omega_{c,1}$, the behavior of the magnitude Bode plot is already defined, but not for the phase Bode plot. In this case we just define the plot to connect $(\omega_{c,1}/10, 0)$ and $(10\omega_{c,1}, -\frac{\pi}{2})$ by a line.

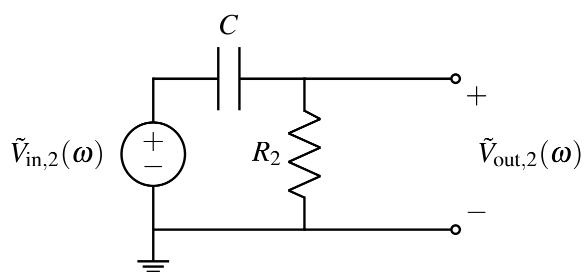
Using these, we can draw the Bode plots.

Plot of $|H_1(\omega)|$.





(b) Consider the second half of the circuit:



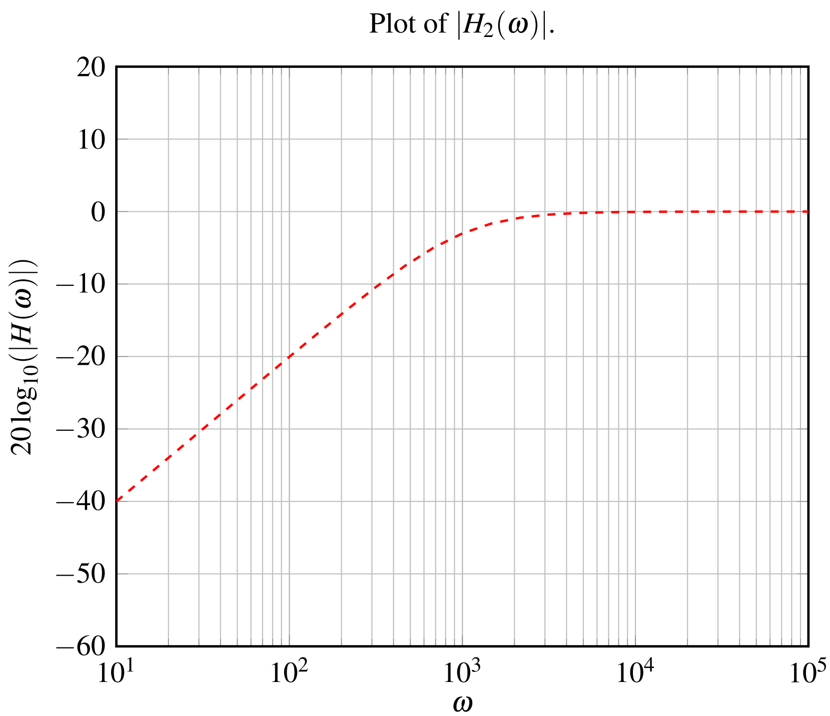
We learned in the previous discussion that the transfer function is given by

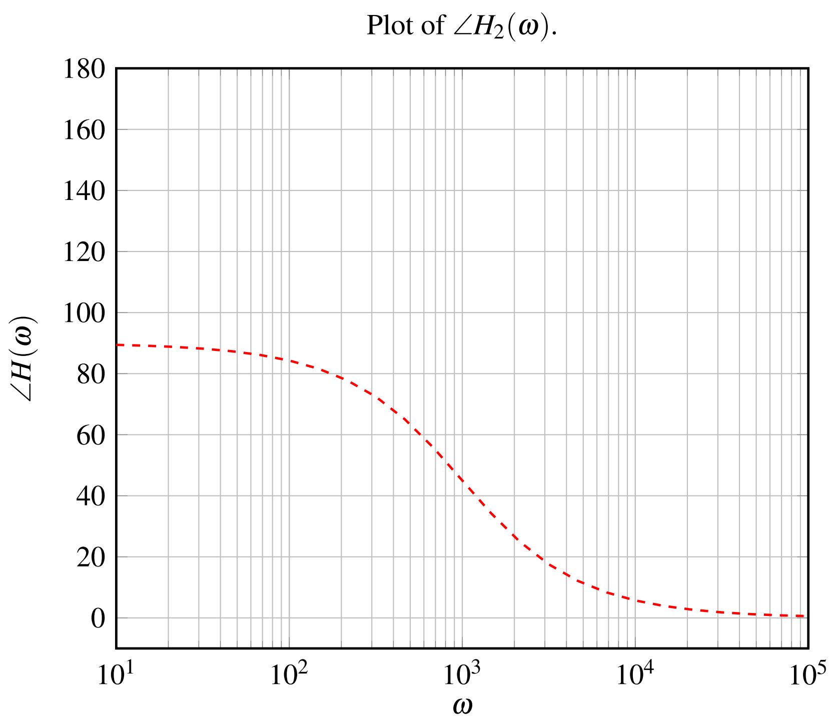
$$H_2(\omega) = \frac{\widetilde{V}_{\text{out},2}}{\widetilde{V}_{\text{in},2}} = \frac{j\omega R_2 C}{1 + j\omega R_2 C},$$

the cutoff frequency $\omega_{c,2}$ is given by

$$\omega_{c,2} = \frac{1}{R_2 C} = \frac{1}{(1\,\text{k}\Omega)\cdot(1\,\mu\text{F})} = 1\times 10^3\,\frac{\text{rad}}{\text{s}},$$

and plots of the transfer function are given by





On these grids, **draw the Bode plots for magnitude and phase**.

Answer: We recognize that we can write $H_2(\omega)$ in the form

$$H_2(\omega) = \frac{j\omega R_2 C}{1 + j\omega R_2 C} = \frac{j\frac{\omega}{\omega_{c,2}}}{1 + j\frac{\omega}{\omega_{c,2}}}.$$

Now we know the "recipe" to draw Bode plots, in particular

- For $\omega \ll \omega_{c,2}$,

$$H_2(\omega) = \frac{j\frac{\omega}{\omega_{c,2}}}{1 + j\frac{\omega}{\omega_{c,2}}} \approx \frac{j\frac{\omega}{\omega_{c,2}}}{1} = j\frac{\omega}{\omega_{c,2}}.$$

  What this means is that

  - For the Bode plot of $\left|H_2(\omega)\right|$ vs. ω:

$$20\log_{10}(\left|H_2(\omega)\right|) \approx 20\log_{10}\left(\frac{\omega}{\omega_{c,2}}\right) = 20\log_{10}(\omega) - 20\log_{10}(\omega_{c,2}).$$

    Correspondingly, in the Bode plot, for $\omega < \omega_{c.2}$, the plot increases with slope 20 per decade.

  - For the Bode plot of $\angle H_2(\omega)$ vs. ω:

$$\angle H_1(\omega) \approx \angle\left(j\frac{\omega}{\omega_{c,2}}\right) = \angle j = \frac{\pi}{2}.$$

    Correspondingly, in the Bode plot, for $\omega < \omega_{c.2}/10$, the plot is constant with $\angle H_1(\omega) = \frac{\pi}{2}$.

- For $\omega \gg \omega_{c,2}$,

$$H_2(\omega) = \frac{j\frac{\omega}{\omega_{c,2}}}{1 + j\frac{\omega}{\omega_{c,2}}} \approx \frac{j\frac{\omega}{\omega_{c,2}}}{j\frac{\omega}{\omega_{c,2}}} = 1.$$

  What this means is that

  - For the Bode plot of $\left|H_2(\omega)\right|$ vs. ω:

$$20\log_{10}(\left|H_2(\omega)\right|) \approx 20\log_{10}(1) = 0.$$

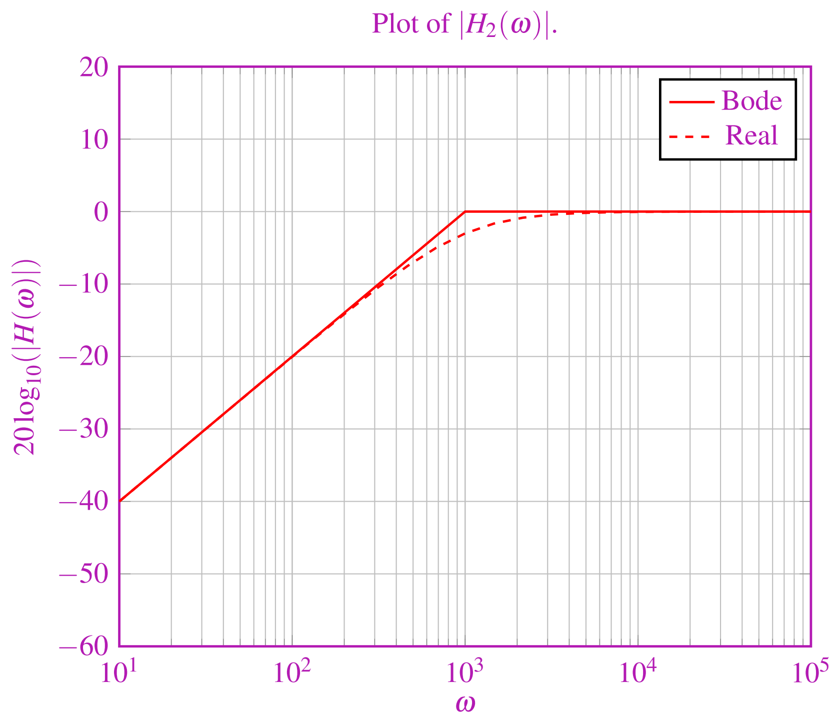
    Correspondingly, in the Bode plot, for $\omega > \omega_{c.2}$, the plot is constant with $20\log_{10}(\left|H_2(\omega)\right|) = 0$.

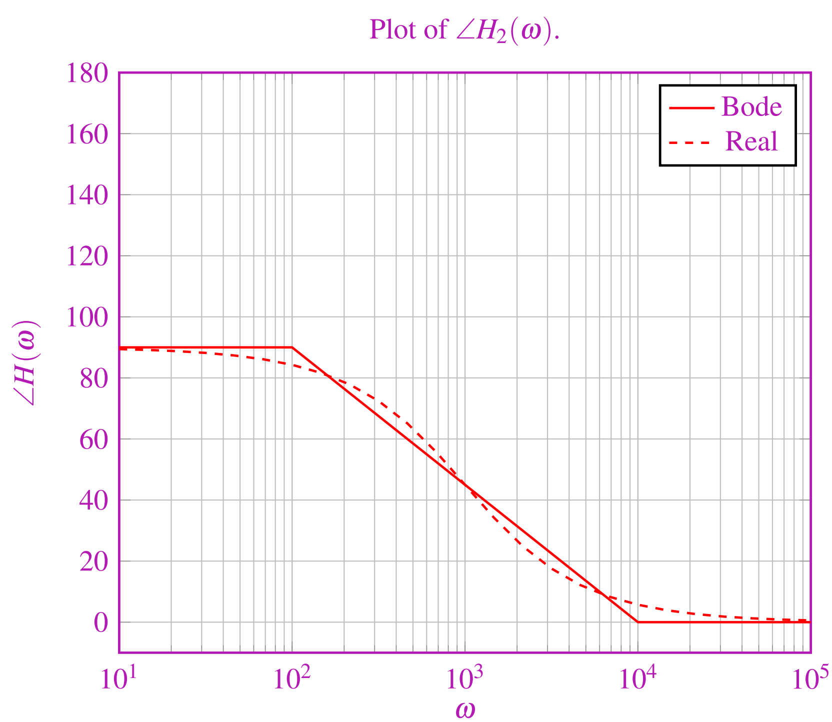
  - For the Bode plot of $\angle H_2(\omega)$ vs. ω:

$$\angle H_2(\omega) \approx \angle 1 = 0.$$

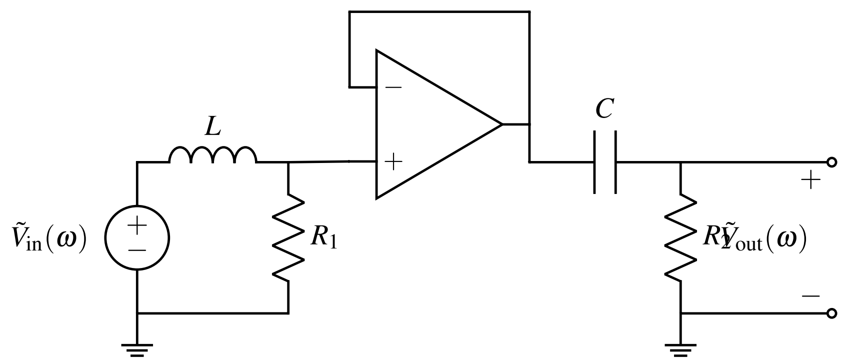
    Correspondingly, in the Bode plot, for $\omega > 10\omega_{c.2}$, the plot is constant with $\angle H_2(\omega) = 0$.

- For ω such that $\omega_{c,2}/10 < \omega < 10\omega_{c,2}$, the behavior of the magnitude Bode plot is already defined, but not for the phase Bode plot. In this case we just define the plot to connect $(\omega_{c,2}/10, \frac{\pi}{2})$ and $(10\omega_{c,2}, 0)$ by a line.





(c) Now, we will put this circuit together. Recall the original diagram:

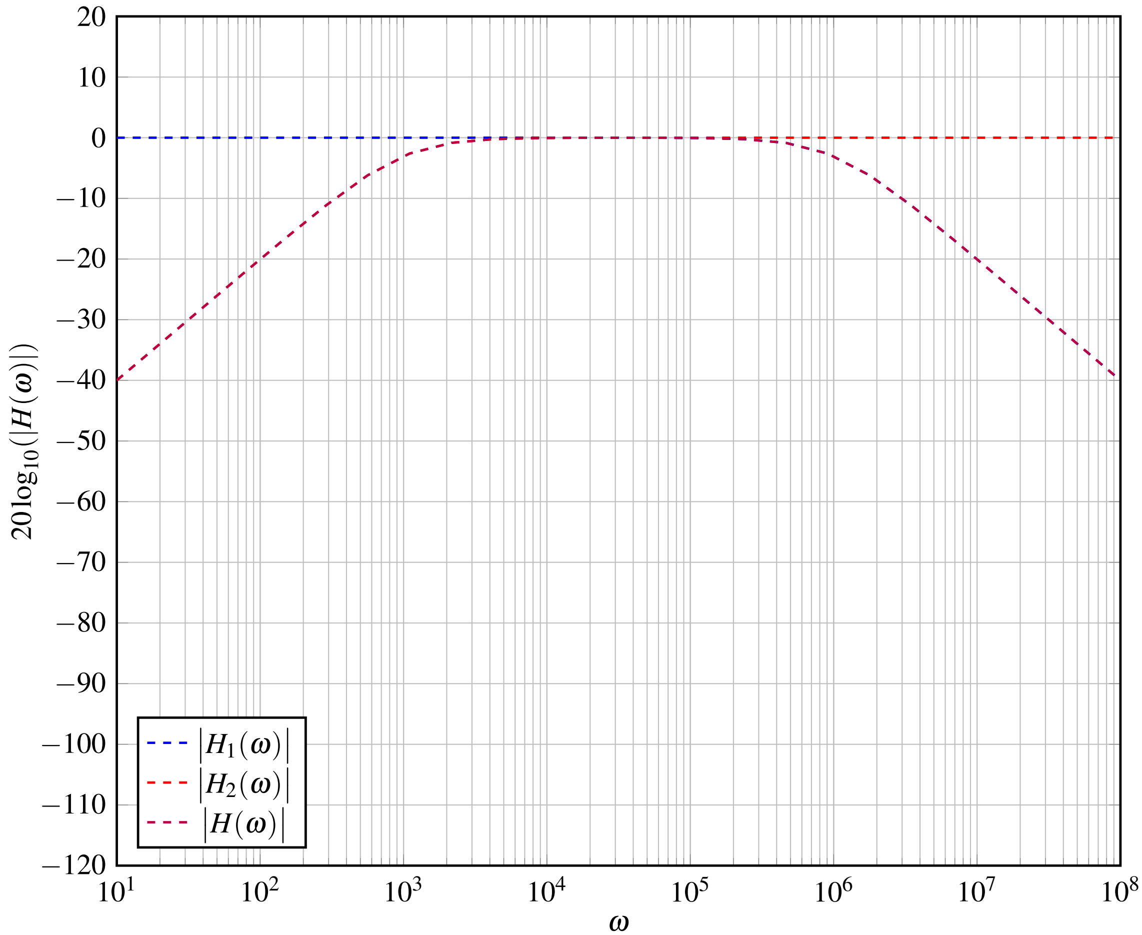


We learned in the previous discussion that the transfer function is

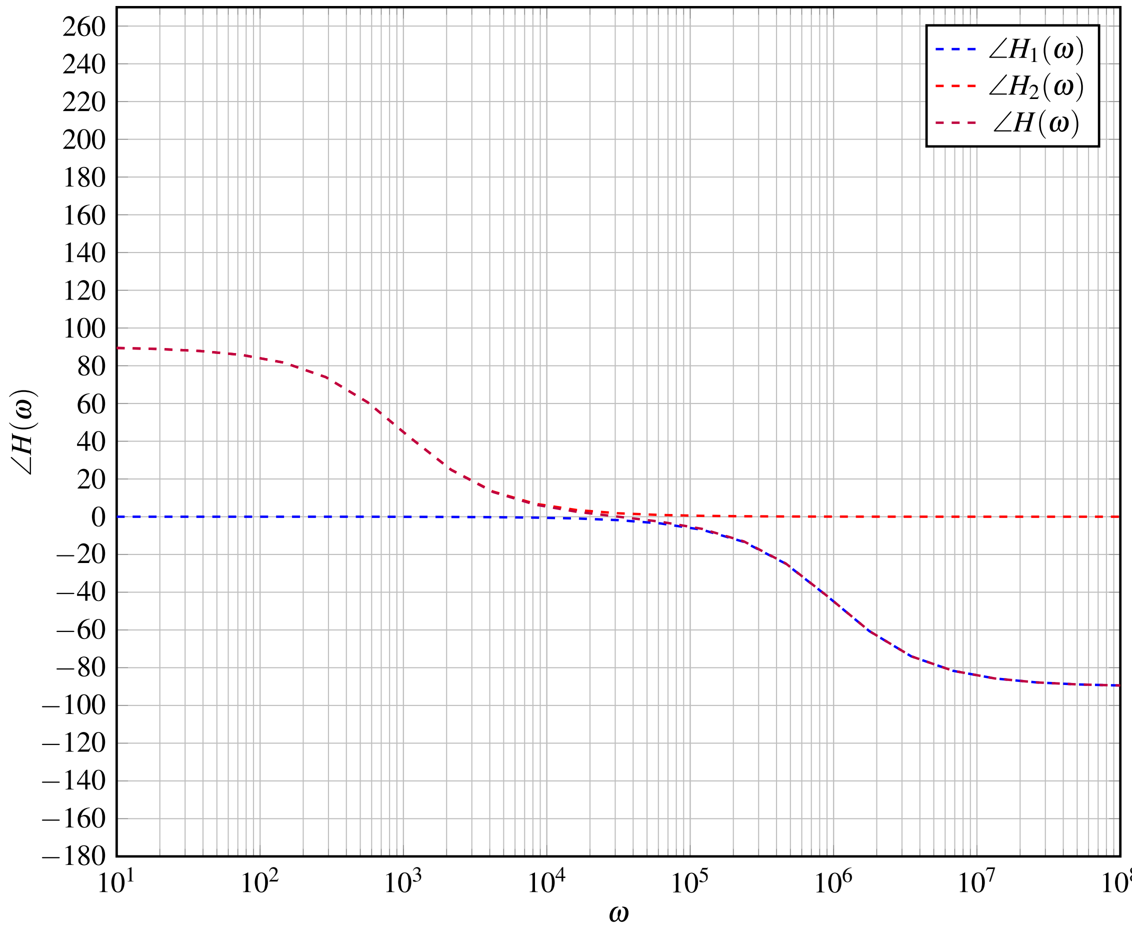
$$H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = H_1(\omega)H_2(\omega)$$

and the transfer function plots are given by

Plot of $|H(\omega)|$.



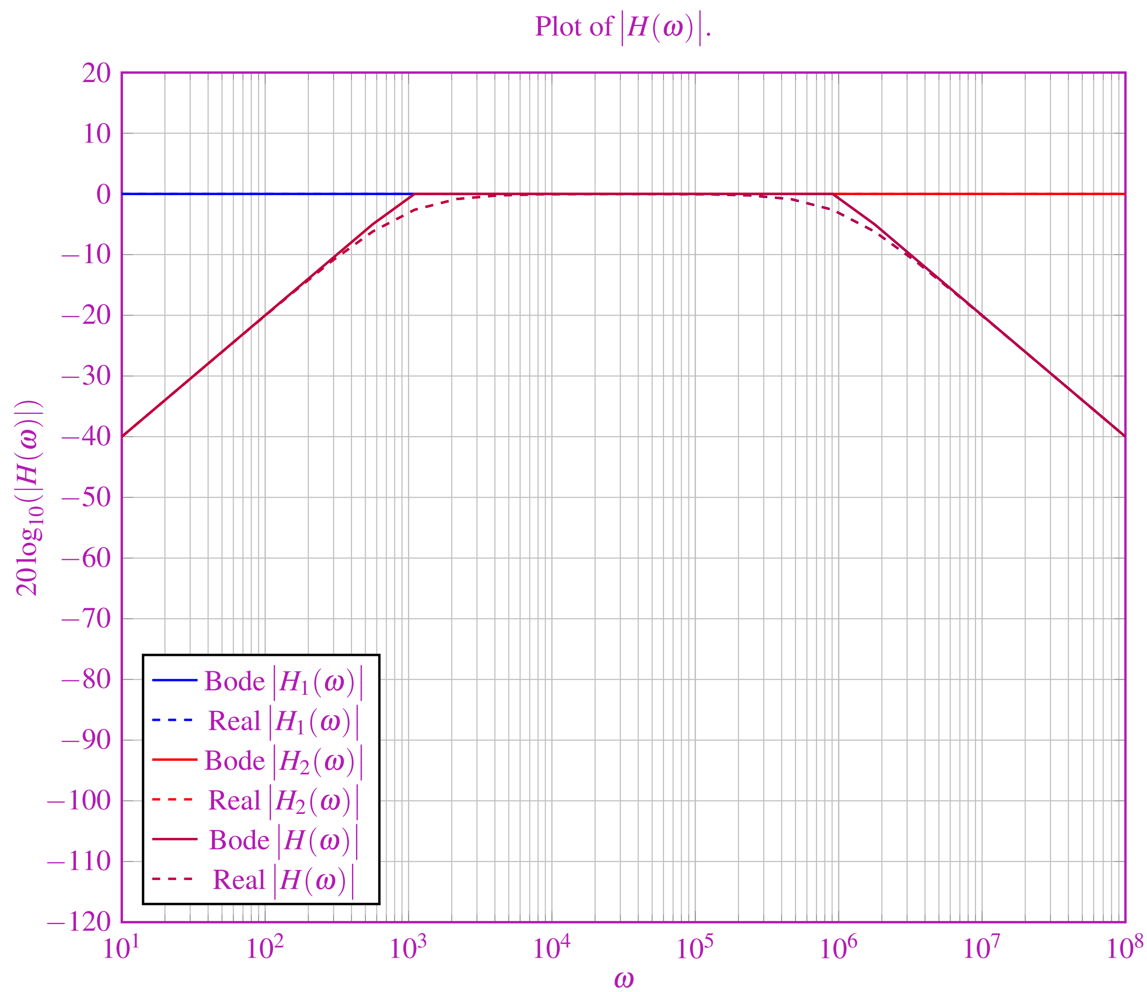
Plot of $\angle H(\omega)$.

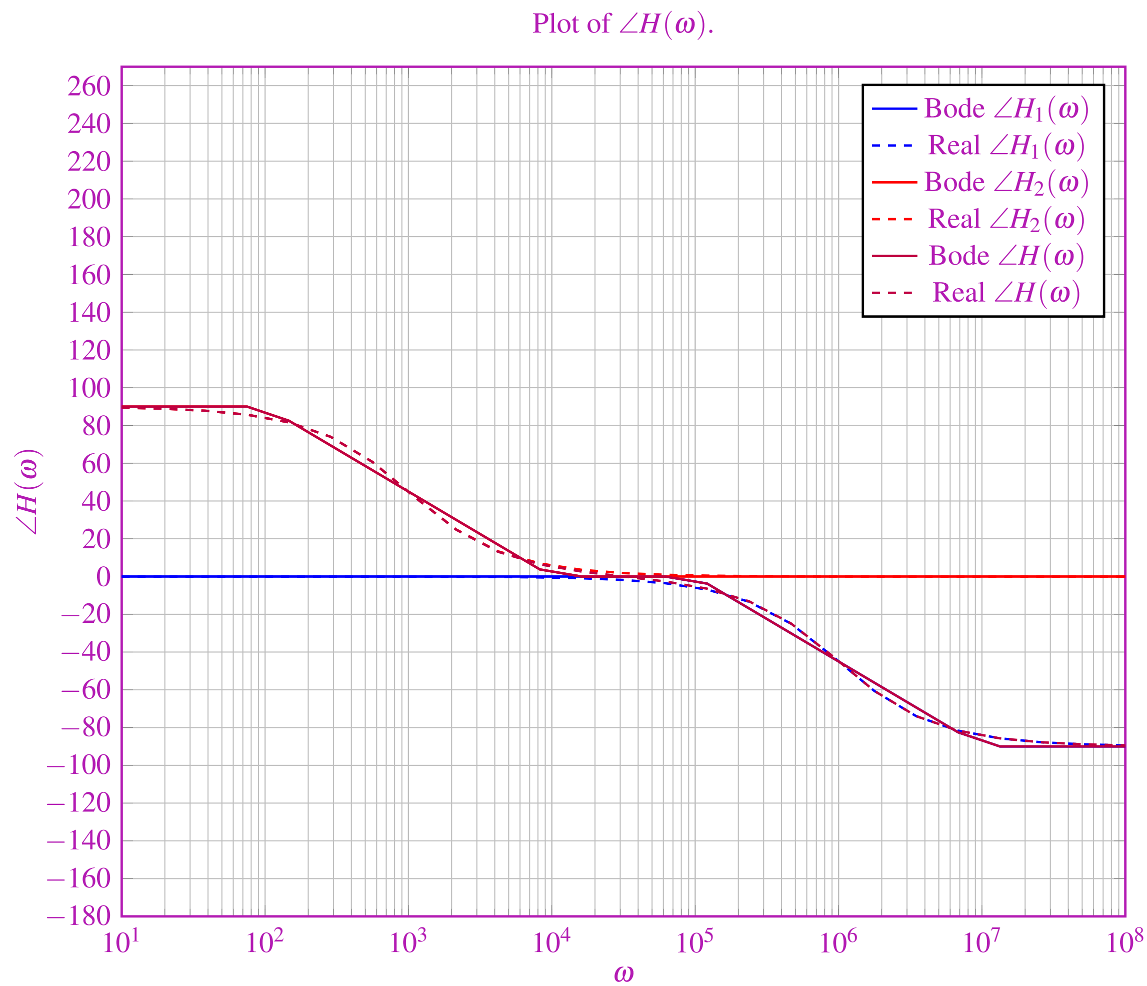


On these grids, **draw the Bode plots for magnitude and phase**.
Hint: Recall that

$$\begin{aligned} 20\log_{10}(|H(\omega)|) &= 20\log_{10}(|H_1(\omega)H_2(\omega)|) = 20\log_{10}(|H_1(\omega)||H_2(\omega)|) \\ &= 20\log_{10}(|H_1(\omega)|) + 20\log_{10}(|H_2(\omega)|) \\ \text{and } \angle H(\omega) &= \angle H_1(\omega) + \angle H_2(\omega). \end{aligned}$$

Answer: What the hint means is that *we can add the plots (for both magnitude and phase) of $H_1(\omega)$ and $H_2(\omega)$ to get the plot for $H(\omega)$*. In general this will let us do analysis of higher-order circuits by breaking them down into easily-analyzable chunks and adding the plots. Of course, since this property holds for the transfer functions, it holds for the Bode plots (which are good linear approximations to the transfer functions) too. So our plots end up looking like this:


Plot of $|H(\omega)|$.
$20\log_{10}(|H(\omega)|)$
20
10
0
−10
−20
−30
−40
−50
−60
−70
−80
−90
−100
−110
−120
10^1
10^2
10^3
10^4
10^5
10^6
10^7
10^8
ω
Bode $|H_1(\omega)|$
Real $|H_1(\omega)|$
Bode $|H_2(\omega)|$
Real $|H_2(\omega)|$
Bode $|H(\omega)|$
Real $|H(\omega)|$



Contributors:

- Sanjit Batra.
- Kris Pister.
- Pavan Bhargava.
- Anant Sahai.
- Druv Pai.