EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet

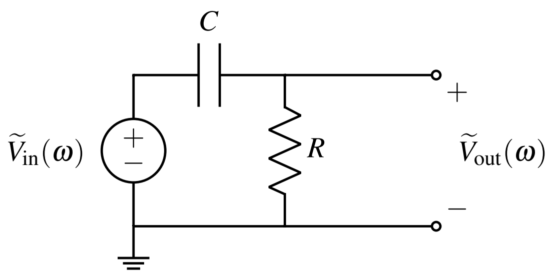
# Discussion 6A

The relevant notes for this discussion are Note 5 and Note 6.

## 1. Transfer function practice

In this problem, you'll be deriving some transfer functions on your own. For each circuit, determine $H(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$. How does each circuit respond as $\omega \to 0$ (low frequencies)? as $\omega \to \infty$ (high frequencies)?

(a)




**Answer:** We'll use the voltage divider formula to find $\widetilde{V}_{\text{out}}$ :

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_R + Z_C}\widetilde{V}_{\text{in}}.$$

Plugging in the impedances gives

$$H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega}{j\omega + \frac{1}{RC}}. \tag{1}$$

Multiplying the numerator and denominator by $j\omega C$ gives

$$H(\omega) = \frac{j\omega RC}{j\omega RC + 1}. \tag{2}$$

An alternate form that is sometimes useful can be derived by dividing out the $RC$ from both numerator and denominator.

$$H(\omega) = \frac{j\omega}{j\omega + \frac{1}{RC}}. \tag{3}$$

At low frequencies, we have

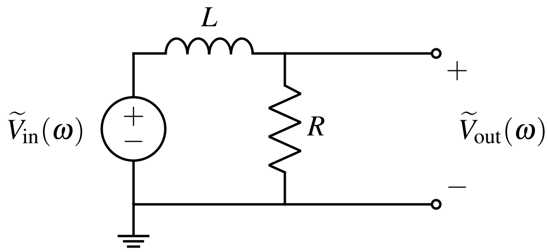
$$\lim_{\omega \to 0} H(\omega) = 0, \tag{4}$$

while at high frequencies, we have

$$\lim_{\omega \to \infty} H(\omega) = 1. \tag{5}$$

So this circuit is a *high-pass filter.*

(b)




**Answer:** The strategy is the same as the previous part, that is, using the voltage divider formula, *i.e.*,

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_R + Z_L}\widetilde{V}_{\text{in}}.$$

A similar manipulation to the previous part gives

$$H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega\frac{L}{R}}. \tag{6}$$

At low frequencies, we have

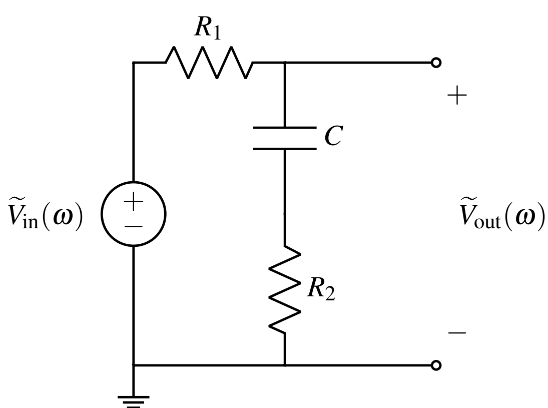
$$\lim_{\omega\to 0} H(\omega) = 1, \tag{7}$$

while at high frequencies, we have

$$\lim_{\omega\to\infty} H(\omega) = 0. \tag{8}$$

So this circuit is a *low-pass filter*. Notice that this circuit resembles the one in the previous part, except we have replaced the capacitor with an inductor. The effect of this change was to reverse the low-frequency and high-frequency behavior of the circuit! Another example of the complementarity of capacitors and inductors.

(c)




**Answer:** Even though there are three components instead of two, we can still use the voltage divider formula by treating $R_2$ and $C$ as a single impedance $Z = R_2 + \frac{1}{j\omega C}$. This would give us

$$\widetilde{V}_{\text{out}} = \frac{Z}{Z_{R_1} + Z}\widetilde{V}_{\text{in}}.$$

Then, the transfer function is

$$H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} \to \frac{R_2 + \frac{1}{j\omega C}}{R_1 + R_2 + j\frac{1}{\omega C}} = \frac{1 + j\omega R_2 C}{1 + j\omega C(R_1 + R_2)}. \tag{9}$$

At low frequencies, we have

$$\lim_{\omega \to 0} H(\omega) = 1, \tag{10}$$

while at high frequencies, we have

$$\lim_{\omega \to \infty} H(\omega) = \frac{R_2}{R_1 + R_2}. \tag{11}$$

So at high frequencies, this circuit behaves like a regular voltage divider with just $R_1$ and $R_2$, as if the capacitor had vanished. This circuit is like a combination of a low-pass filter and a voltage divider: low frequency inputs are preserved, and high-frequency signals are diminished.

**2. Plotting and combining transfer functions**

Recall that any transfer function can be written in polar form as

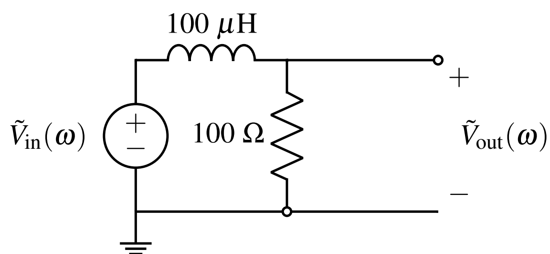
$$H(\omega) = A(\omega)e^{j\alpha(\omega)}$$

where $A(\omega)$ and $\alpha(\omega)$ are real functions of omega giving the magnitude and phase of the transfer function, respectively. To see how transfer functions combine, consider two $H_1(\omega)$ and $H_2(\omega)$.

$$\begin{aligned} H_1(\omega) &= Ae^{j\alpha} \\ H_2(\omega) &= Be^{j\beta} \\ H_1(\omega)\cdot H_2(\omega) &= Ae^{j\alpha}Be^{j\beta} = ABe^{j(\alpha+\beta)} \\ \frac{H_1(\omega)}{H_2(\omega)} &= \frac{Ae^{j\alpha}}{Be^{j\beta}} = \frac{A}{B}e^{j(\alpha-\beta)} \end{aligned}$$

As you can see, magnitudes of transfer functions multiply and divide while the phases add and subtract.

In this problem we will try to plot the transfer function of the following circuit:




(a) Write expressions for $|H(\omega)|$ and $\angle H(\omega)$. For now, you can keep it in terms of $R$ and $L$.

**Answer:** We already found the transfer function in the previous exercise: specifically,

$$H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega\frac{L}{R}}. \tag{12}$$

The magnitude can be found by dividing the magnitueds of the numerator and denominator:

$$|H(\omega)| = \frac{|1|}{\left|1 + j\omega\frac{L}{R}\right|} \tag{13}$$

$$= \frac{1}{\sqrt{1 + \omega^2\frac{L^2}{R^2}}}. \tag{14}$$

Similarly the phase can be found by subtracting the phases of the numerator and denominator

$$\angle H(\omega) = \angle(1) - \angle\left(1 + j\omega\frac{L}{R}\right) \tag{15}$$

$$= 0 - \text{atan2}\left(\omega\frac{L}{R}, 1\right). \tag{16}$$

(b) What is the cutoff frequency for this circuit? Mark it on the log-log plot with a vertical line. Recall that a transfer function of the form $H(\omega) = \frac{k}{1+j\omega/\omega_c}$ has a cutoff frequency of $\omega_c$.

**Answer:** In this case, it will be the inverse of the LR time constant, that is

$$\omega_c = \frac{R}{L}. \tag{17}$$

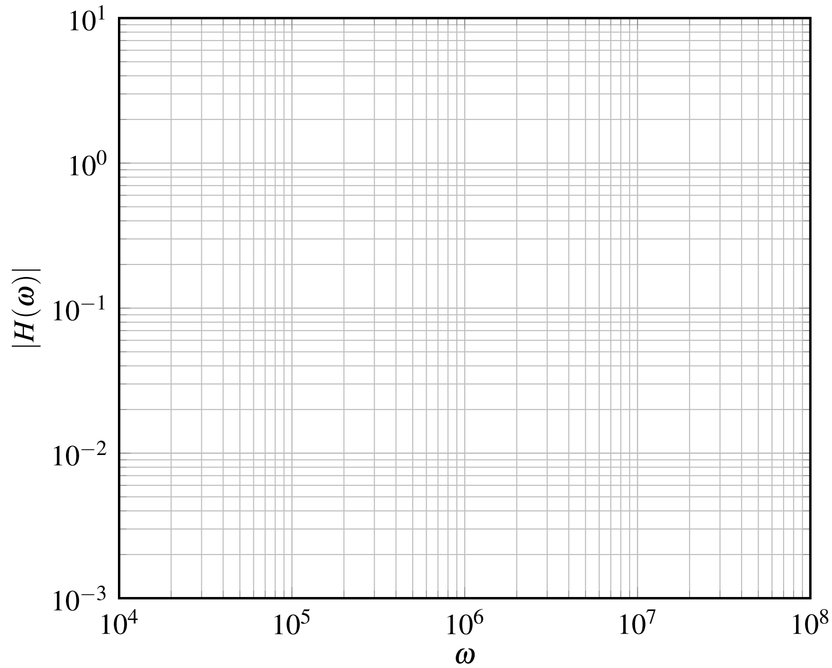
For our given values, that's

$$\omega_c = \frac{100\ \Omega}{100\ \mu\text{H}} = 10^6 \text{ radians/s.} \tag{18}$$

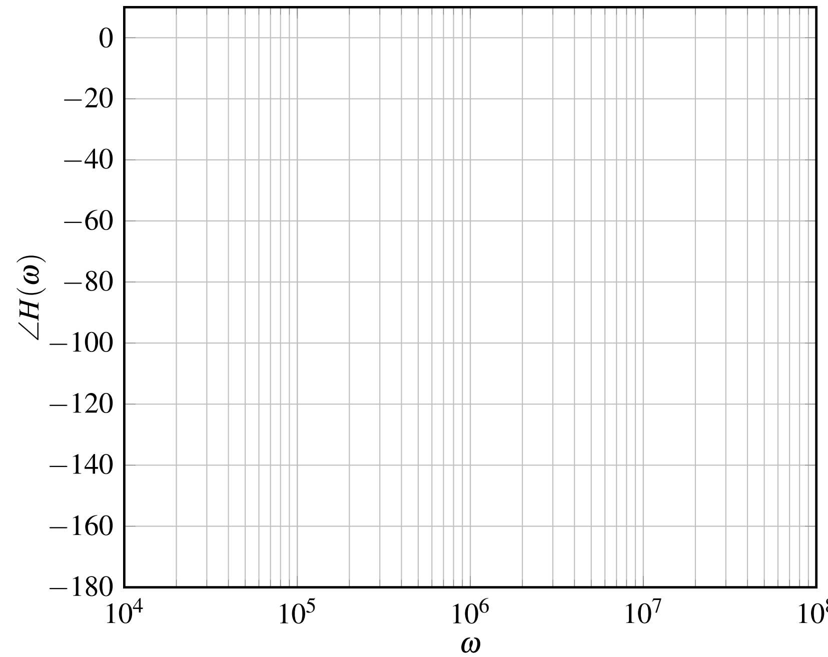
(c) Sketch plots of the magnitude and phase of this transfer function. We have provided a table with the transfer function evaluated at a few representative points around the cutoff frequency.

| $\omega$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|---|---|---|---|---|---|
| $\lvert H(\omega)\rvert$ | 1.00 | 0.995 | 0.707 | 0.100 | 0.01 |
| $\angle H(\omega)$ | -0.6 | -6 | -45 | -84 | -89 |

Plot of $|H(\omega)|$ (for **you** to draw).

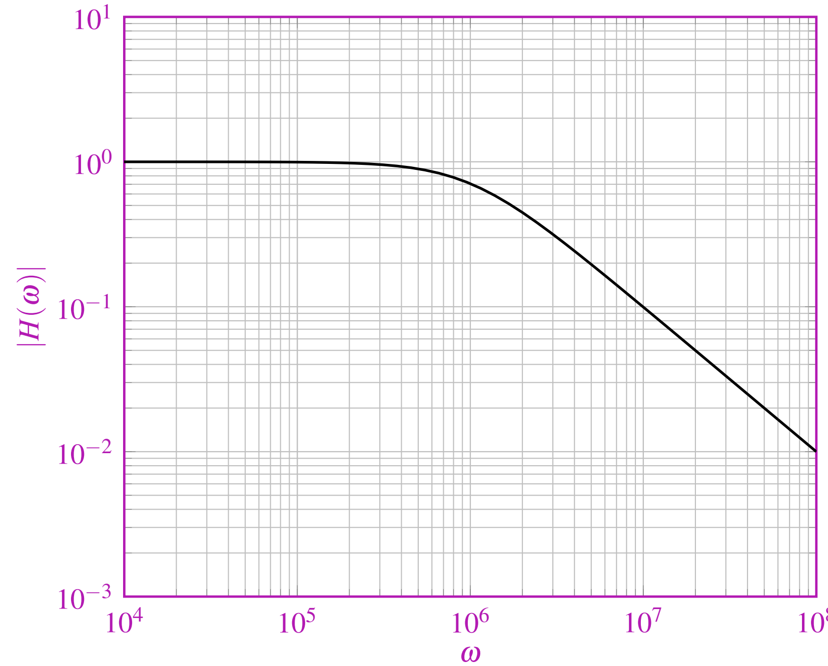


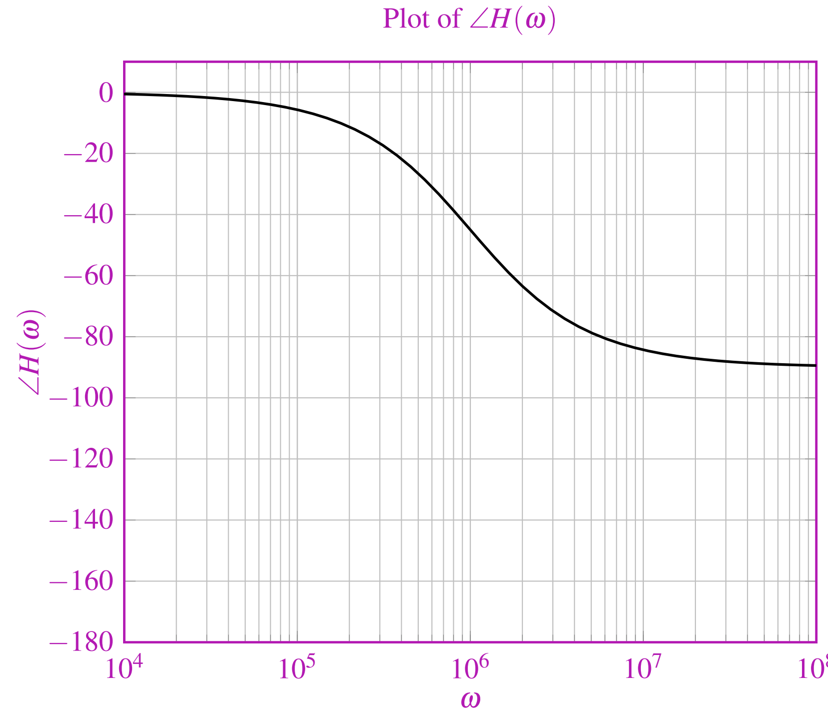
Plot of $\angle H(\omega)$ (for **you** to draw).




**Answer:** The final drawings should look like this:

Plot of $|H(\omega)|$





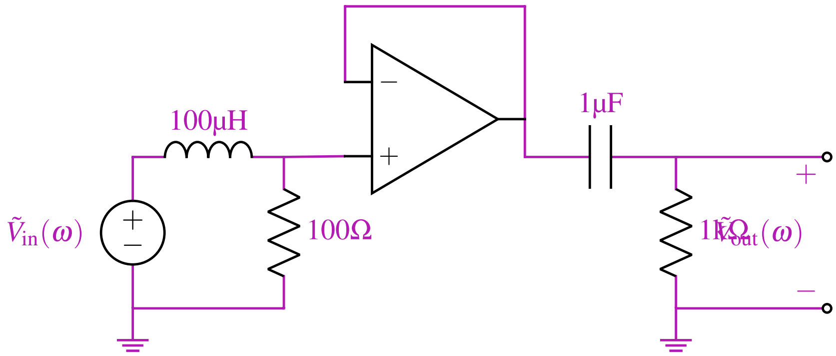
Plot of $\angle H(\omega)$
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
$\angle H(\omega)$
$10^4$
$10^5$
$10^6$
$10^7$
$10^8$
$\omega$

(d) Now suppose we want to compose the filter from part (1.a) with this new circuit. Use $R = 1\text{k}\Omega$ and $C = 1\mu\text{F}$ for the filter from part (1.a). We can compose two circuits by connecting the output of the first circuit into the second circuit, through a unity gain buffer. For this problem, call the circuit from this system with an inductor $H_1$, and the filter from (1.a) $H_2$. The transfer function of the composed circuit is:

$$H(\omega) = H_1(\omega) \cdot H_2(\omega)$$

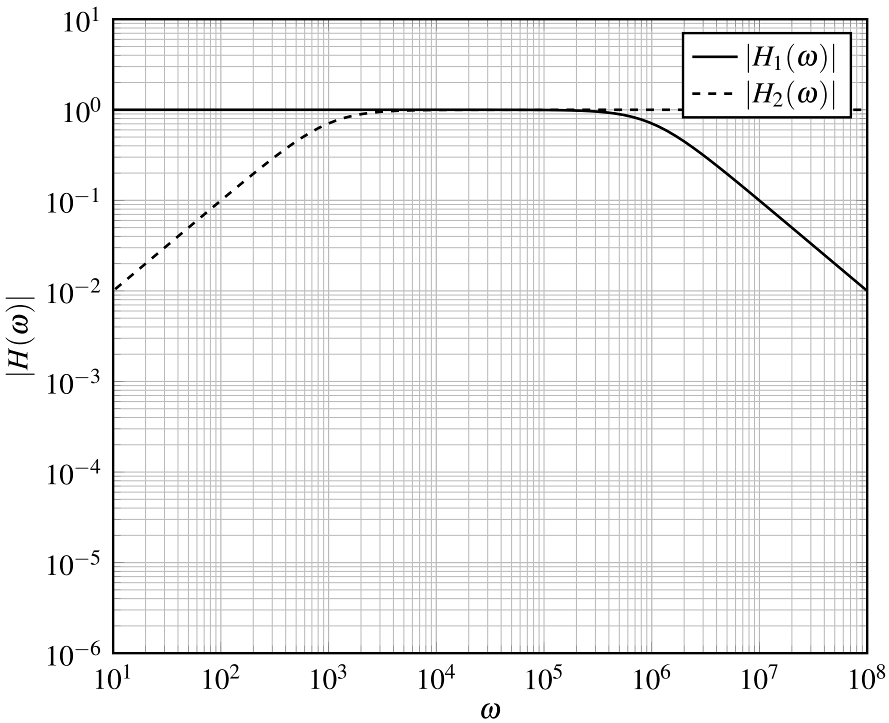
i. Draw this circuit.

**Answer:**



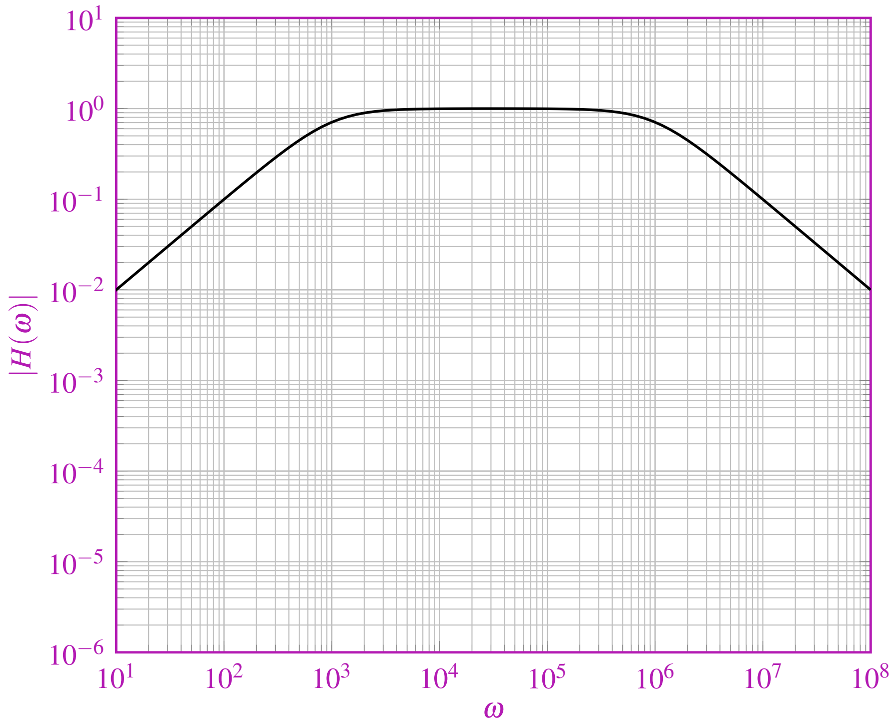

ii. Plot the magnitude of the composed circuit. Below is a log-log plot with the magnitudes of $|H_1(\omega)|$ and $|H_2(\omega)|$ drawn to assist you. Recall that $|AB| = |A| \cdot |B|$ for complex numbers $A$ and $B$.

Plots of $|H_i(\omega)|$.



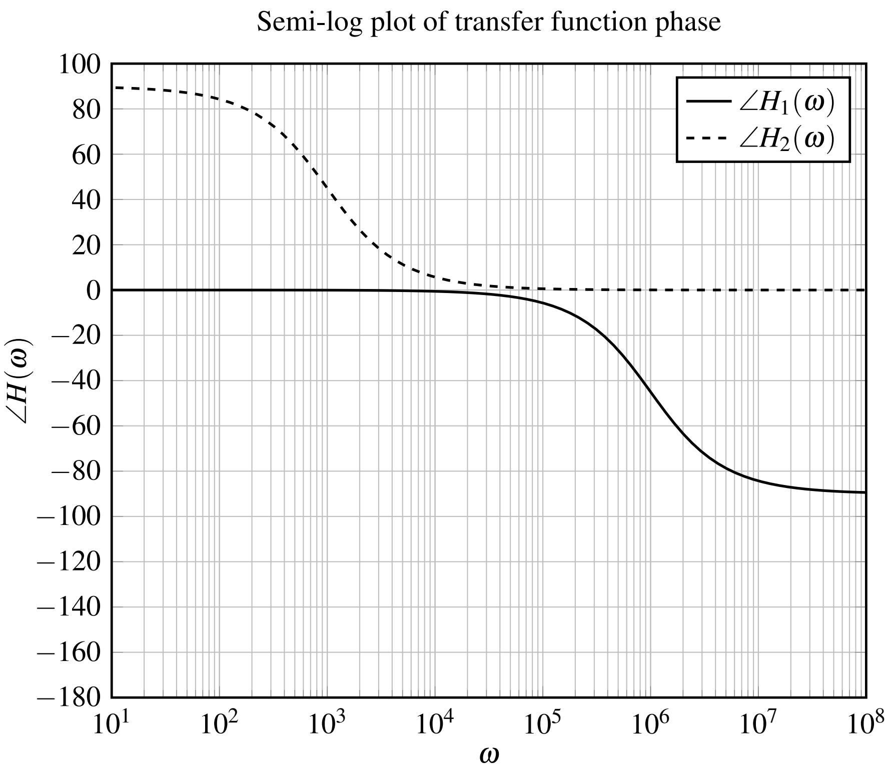
**Answer:** The final magnitude plot should be constructed by plotting the products of the two lines at each frequency.

Solution plot of transfer function magnitude

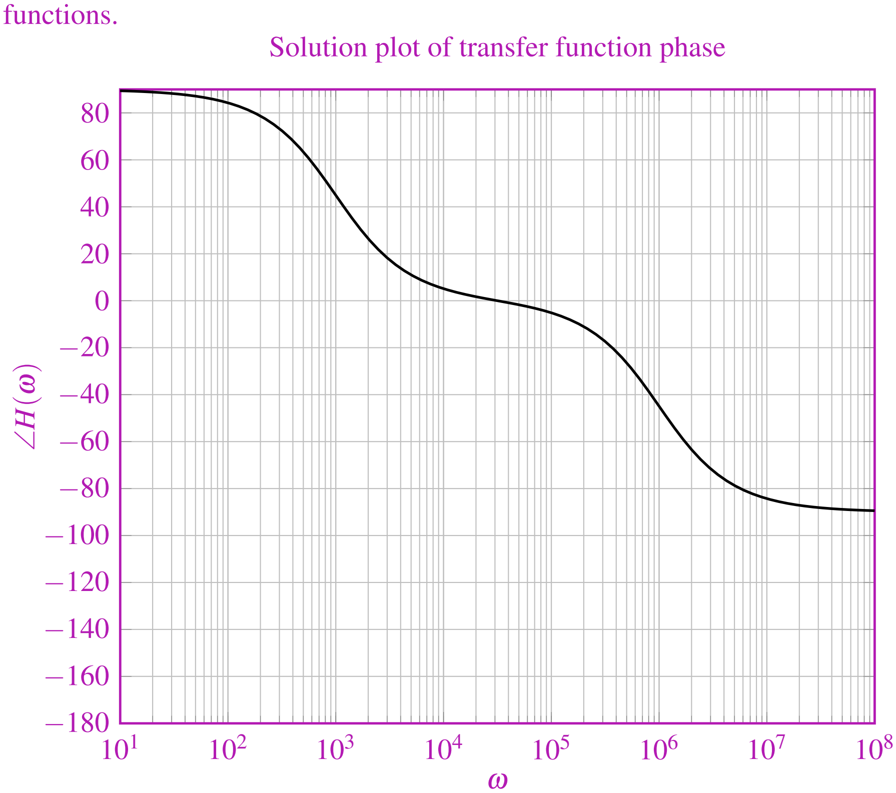



This is a band-pass filter.

ii. Plot the phase of the composed circuit. Below is a semi-log plot with the phases $\angle H_1(\omega)$ and $\angle H_2(\omega)$ drawn to assist you.




**Answer:** The final phase plot should be obtained by adding the lines from the two transfer functions.

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