EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet — Discussion 5B

For this discussion, Note 4 is helpful.

Some equations which may be useful are:

- $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$ for all real numbers x.
- $\cos(x) = \frac{1}{2}(e^{jx} + e^{-jx})$ for all real numbers x.
- The impedance of a resistor with resistance R is $Z_R = R$.
- The impedance of a capacitor with capacitance C is $Z_C = \frac{1}{j\omega C}$.
- The impedance of an inductor with inductance L is $Z_L = j\omega L$.

1. Phasor Analysis

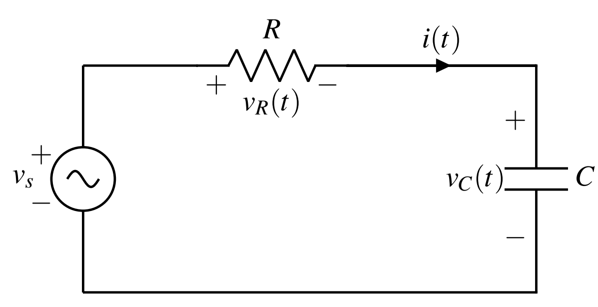
Any sinusoidal time-varying function $x(t)$, representing a voltage or a current, can be expressed in the form

$$x(t) = \widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}, \tag{1}$$

where $\widetilde{X}$ is a time-independent function called the **phasor** representation of $x(t)$ (recall that $\overline{a}$ denotes the complex conjugate of a). Note that 1) $\widetilde{X}$ and $\overline{\widetilde{X}}$ are complex conjugates of each other, 2) $e^{j\omega t}$ and $e^{-j\omega t}$ are complex conjugates of each other, and 3) that $\widetilde{X}e^{j\omega t}$ and $\overline{\widetilde{X}}e^{-j\omega t}$ are also complex conjugates of each other.

Note: We define the phasor as the coefficient of $e^{j\omega t}$ in eq. (1). Other resources (such as some past iterations of this class) define it slightly differently; the definitions differ by a constant multiple. Some reasons for competing definitions are discussed in Note 4. Although the definitions in general lead to the same answers, be careful to use our class' definition and not get tripped up. For example, if we ask about the magnitude of the phasor, you wouldn't want to be off by a constant!

The phasor analysis method consists of five steps. Consider the RC circuit below.



The voltage source is given by

$$v_s(t) = 12\sin\left(\omega t - \frac{\pi}{4}\right), \tag{2}$$

with $\omega = 1 \times 10^3 \, \frac{\text{rad}}{\text{s}}$, $R = \sqrt{3}\text{k}\Omega$, and $C = 1\,\mu\text{F}$.

Our goal is to obtain a solution for $i(t)$ with the sinusoidal voltage source $v_s(t)$.

(a) **Step 1: Write sources as exponentials:** $\widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}$

All voltages and currents with known sinusoidal functions should be expressed in the standard exponential format. Convert $v_s(t)$ into a exponential and write down its phasor representation $\widetilde{V}_s$.

Answer:

$$\begin{aligned}
v_s(t) &= 12\sin\left(\omega t - \frac{\pi}{4}\right) \\
&= 12\cos\left(\omega t - \frac{3\pi}{4}\right) \\
&= 12\left(\frac{e^{j(\omega t - 3\pi/4)}}{2} + \frac{e^{-j(\omega t - 3\pi/4)}}{2}\right) \\
&= 12\left(\frac{e^{-j(3\pi/4)}}{2}e^{j\omega t} + \frac{e^{j(3\pi/4)}}{2}e^{-j\omega t}\right) \\
&= 6e^{-j(3\pi/4)}e^{j\omega t} + 6e^{j(3\pi/4)}e^{-j\omega t} \\
&= 6e^{-j(3\pi/4)}e^{j\omega t} + \overline{6e^{-j(3\pi/4)}}e^{-j\omega t} \\
&= \widetilde{V}_s e^{j\omega t} + \overline{\widetilde{V}}_s e^{-j\omega t}
\end{aligned}$$

From the problem statement and pattern matching, we can extract the phasor representation as the coefficient of $e^{j\omega t}$:

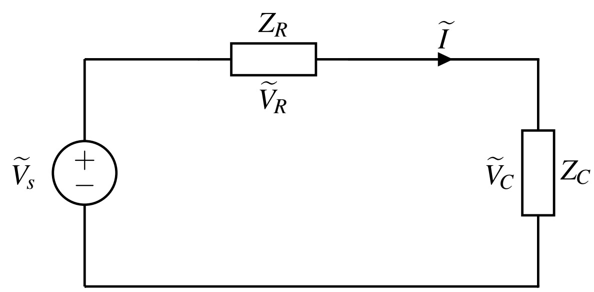
$$\widetilde{V}_s = 6e^{-j\frac{3\pi}{4}}$$

(b) **Step 2: Transform circuits to phasor domain**

The voltage source is represented by its phasor $\widetilde{V}_s$. The current $i(t)$ is related to its phasor counterpart $\widetilde{I}$ by

$$i(t) = \widetilde{I}e^{j\omega t} + \overline{\widetilde{I}}e^{-j\omega t}.$$

We redraw the circuit in phasor domain:



What are the impedances of the resistor, Z_R, and capacitor, Z_C? We sometimes also refer to this as the "phasor representation" of R and C.

Answer:

$$Z_R = R$$
$$Z_C = \frac{1}{j\omega C}$$

(c) **Step 3: Cast the branch and element equations in phasor domain**
Use Kirchhoff's laws to write down a loop equation that relates all phasors in Step 2.

Answer:
The KVL equations for this circuit are:

$$v_s(t) = v_R(t) + v_C(t),$$

where we have denoted the voltage across the resistor as $v_R(t)$. If you expand all these quantities using phasors (using Equation (1)), we get

$$\underbrace{\widetilde{V}_s e^{j\omega t} + \overline{\widetilde{V}_s} e^{-j\omega t}}_{v_s(t)} = \underbrace{\widetilde{V}_R e^{j\omega t} + \overline{\widetilde{V}_R} e^{-j\omega t}}_{v_R(t)} + \underbrace{\widetilde{V}_C e^{j\omega t} + \overline{\widetilde{V}_C} e^{-j\omega t}}_{v_C(t)}$$

Collecting together all the $e^{j\omega t}$ terms and all the $e^{-j\omega t}$ terms, the above equation can be rewritten as

$$(\widetilde{V}_s - \widetilde{V}_R - \widetilde{V}_C)e^{j\omega t} + (\overline{\widetilde{V}_s - \widetilde{V}_R - \widetilde{V}_C})e^{-j\omega t} = 0.$$

If $\omega \neq 0$,[1] it can be shown[2] that both of the coefficients of $e^{j\omega t}$ and $e^{-j\omega t}$ in the above equation must be equal to 0 for this equation to hold. That is:

$$\widetilde{V}_s - \widetilde{V}_R - \widetilde{V}_C = 0, \quad \text{and}$$
$$\overline{\widetilde{V}_s - \widetilde{V}_R - \widetilde{V}_C} = 0.$$

Both of the equations above have the same meaning, *i.e.*, $\widetilde{V}_s - \widetilde{V}_R - \widetilde{V}_C = 0$ or

$$\widetilde{V}_s = \widetilde{V}_R + \widetilde{V}_C.$$

This is exactly the same KVL equation as given by $v_s(t) = v_R(t) + v_C(t)$, but using phasors. In the same way, you can show that KCL is also obeyed by phasors. This conclusion implies that the standard rules for putting together circuit equations using KCL and KVL work identically with phasors as with time-varying notation. The only difference is that the voltage-current relationships of elements should be in phasor form, *e.g.*, $\widetilde{I} = j\omega C\widetilde{V}_C = \frac{1}{Z_C}\widetilde{V}_C$.
We can apply what we've found above to write the circuit in the phasor domain:

$$Z_R\widetilde{I} + Z_C\widetilde{I} = \widetilde{V}_s$$
$$\left(R + \frac{1}{j\omega C}\right)\widetilde{I} = 6e^{-j\frac{3\pi}{4}}$$

Now that we've shown that the phasor representation (*i.e.*, $\widetilde{I}$ and $\widetilde{V}$) of our circuit is equivalent to the time-varying representation (*i.e.*, $i(t)$ and $v(t)$), in the future we can write the KCL and KVL equations in phasor form directly.

[1] Try working out the $\omega = 0$ case by yourself! It's even easier.
[2] This can be shown because the functions $e^{j\omega t}$ and $e^{-j\omega t}$ are linearly independent.

(d) **Step 4: Solve for unknown variables**

Solve the equation you derived in Step 3 for $\widetilde{I}$ and $\widetilde{V}_C$. What is the polar form of $\widetilde{I}$ and $\widetilde{V}_C$? Polar form is given by $Ae^{j\theta}$, where A is a positive real number.

Answer:

$$\widetilde{I} = \frac{6e^{-j\frac{3\pi}{4}}}{R + \frac{1}{j\omega C}} = \frac{j \cdot 6\omega C e^{-j\frac{3\pi}{4}}}{j\omega RC + 1}$$

$$\widetilde{V}_C = \widetilde{I} Z_C = \frac{j \cdot 6\omega C e^{-j\frac{3\pi}{4}}}{1 + j\omega RC} \cdot \frac{1}{j\omega C} = \frac{6e^{-j\frac{3\pi}{4}}}{1 + j\omega RC}$$

To derive the polar form, we plug in for the values of the circuit elements:

$$\widetilde{I} = \frac{j \cdot 6\omega C e^{-j\frac{3\pi}{4}}}{1 + j\omega RC} = \frac{j \cdot 6 \cdot 10^3 \cdot 10^{-6} \cdot e^{-j\frac{3\pi}{4}}}{1 + j \cdot 10^3 \cdot \sqrt{3} \cdot 10^3 \cdot 10^{-6}} = \frac{(e^{j\frac{\pi}{2}}) \cdot 6 \cdot 10^{-3} \cdot e^{-j\frac{3\pi}{4}}}{1 + j\sqrt{3}}$$
$$= \frac{6 \cdot 10^{-3} \cdot e^{-j\frac{3\pi}{4}} e^{j\frac{\pi}{2}}}{2e^{j\frac{\pi}{3}}} = 3e^{-j\frac{7\pi}{12}} \text{mA}.$$

$$\widetilde{V}_C = \frac{6e^{-j\frac{3\pi}{4}}}{1 + j\omega RC} = \frac{6e^{-j\frac{3\pi}{4}}}{1 + j\sqrt{3}} = \frac{6e^{-j\frac{3\pi}{4}}}{2e^{j\frac{\pi}{3}}} = 3e^{-j\frac{13\pi}{12}} \text{V}$$

(e) **Step 5: Transform solutions back to time domain**

To return to time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart. What is $i(t)$ and $v_C(t)$? What is the phase difference between $i(t)$ and $v_C(t)$?

Answer:

$$i(t) = \widetilde{I} e^{j\omega t} + \overline{\widetilde{I}} e^{-j\omega t} = 3e^{-j\frac{7\pi}{12}} e^{j\omega t} + 3e^{j\frac{7\pi}{12}} e^{-j\omega t} = 6\cos\left(\omega t - \frac{7\pi}{12}\right) \text{mA}$$

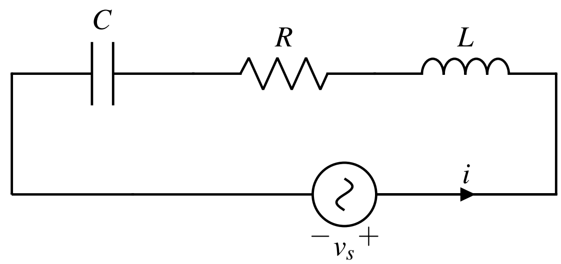
$$v_C(t) = \widetilde{V}_C e^{j\omega t} + \overline{\widetilde{V}_C} e^{-j\omega t} = 3e^{-j\frac{13\pi}{12}} e^{j\omega t} + 3e^{j\frac{13\pi}{12}} e^{-j\omega t} = 6\cos\left(\omega t - \frac{13\pi}{12}\right) \text{V}$$

The phase difference between $i(t)$ and $v_C(t)$ is $\angle\widetilde{I} - \angle\widetilde{V}_C = -\frac{7\pi}{12} - (-\frac{13\pi}{12}) = \frac{\pi}{2}$.

2. RLC Circuit Phasor Analysis

We study a simple RLC circuit with an AC voltage source given by

$$v_s(t) = B\cos(\omega t - \phi)$$

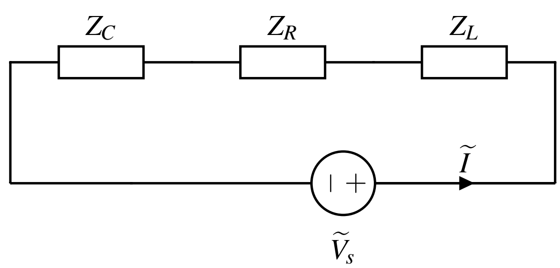


(a) Write out the phasor representation of $v_s(t)$, and the impedances of R, C, and L.

Answer: Using the same derivations as the previous problem's (a) and (b) parts,

$$\widetilde{V}_s = \frac{B}{2}e^{-j\phi}, \quad Z_R = R, \quad Z_C = \frac{1}{j\omega C}, \quad Z_L = j\omega L$$

(b) Now, we're going to redraw the circuit in the phasor domain.



Use Kirchhoff's laws to write down a loop equation relating the phasors.

Answer:

$$Z_R\widetilde{I} + Z_C\widetilde{I} + Z_L\widetilde{I} = \widetilde{V}_s$$
$$\left(R + \frac{1}{j\omega C} + j\omega L\right)\widetilde{I} = \frac{B}{2}e^{-j\phi}$$

(c) Solve the equation in the previous step for the current $\widetilde{I}$. What is the magnitude and phase of the polar form of $\widetilde{I}$?

Hint: You'll need the following identities, which you can find in Note j:

- $\left|z_1/z_2\right| = |z_1|/|z_2|$
- $\angle(\frac{z_1}{z_2}) = \angle z_1 - \angle z_2$
- $\angle(a + jb) = \text{atan2}(b, a)$[3]

[3] We use $\text{atan2}(b,a)$ because passing in $\frac{b}{a}$ to the $\tan^{-1}$ or arctangent function loses information about the signs of b and a, and thus the location of $a + jb$. This is the reason $\tan^{-1}$ outputs numbers only in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$; on the other hand, atan2 outputs numbers in the range $[0, 2\pi]$, which is what we want. More information about atan2 is found in Note j.

Answer:

$$\widetilde{I} = \frac{\frac{B}{2}e^{-j\phi}}{R + \frac{1}{j\omega C} + j\omega L} = \frac{\frac{B}{2}e^{-j\phi}}{R + j(\omega L - \frac{1}{\omega C})}$$

Note that $\widetilde{I}$ is a complex number, more specifically a quotient of complex numbers, with numerator in polar form ($re^{j\theta}$) and denominator in "rectangular" form ($a + jb$). As such, the magnitude of $\widetilde{I}$ is

$$|\widetilde{I}| = \frac{\left|\frac{B}{2}e^{-j\phi}\right|}{\left|R + j(\omega L - \frac{1}{\omega C})\right|} = \frac{\frac{B}{2}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$$

The phase of $\widetilde{I}$ is

$$\angle\widetilde{I} = \angle\left(\frac{B}{2}e^{-j\phi}\right) - \angle\left(R + j\left(\omega L - \frac{1}{\omega C}\right)\right) = -\phi - \operatorname{atan2}\left(\omega L - \frac{1}{\omega C}, R\right).$$

3. (Practice) Inductor Impedance

Given the voltage-current relationship of an inductor $v(t) = L\frac{di(t)}{dt}$, show that its complex impedance is $Z_L = j\omega L$.

Answer:

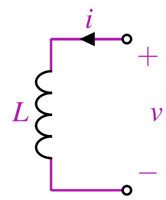


Figure 1: A simple inductor circuit

Consider a simple inductor circuit as in Figure 1, with current being

$$i(t) = I_0 \cos(\omega t + \phi) = \frac{I_0 e^{j\phi}}{2} e^{j\omega t} + \frac{I_0 e^{-j\phi}}{2} e^{-j\omega t}$$

so it has the phasor

$$\widetilde{I} = \frac{I_0 e^{j\phi}}{2}.$$

By the inductor equation,

$$\begin{aligned} v(t) &= L\frac{di(t)}{dt} \\ &= L\frac{d}{dt}\left(\frac{I_0 e^{j\phi}}{2} e^{j\omega t} + \frac{I_0 e^{-j\phi}}{2} e^{-j\omega t}\right) \\ &= \frac{j\omega L I_0 e^{j\phi}}{2} e^{j\omega t} + \frac{-j\omega L I_0 e^{-j\phi}}{2} e^{-j\omega t} \\ &= j\omega L\widetilde{I} e^{j\omega t} + \overline{j\omega L\widetilde{I}} e^{-j\omega t} \\ &= \widetilde{V} e^{j\omega t} + \overline{\widetilde{V}} e^{-j\omega t}. \end{aligned}$$

By pattern matching with the last two equations,

$$\widetilde{V} = j\omega L\widetilde{I}.$$

The impedance of an inductor is an abstraction to model the inductor as a resistor in the phasor domain. This is denoted Z_L.

$$Z_L = \frac{\widetilde{V}}{\widetilde{I}} = j\omega L$$

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