EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet

Discussion 4B

In this discussion, we're going to cover linear algebra concepts (Change of Basis, Diagonalization) that unlock powerful circuit analysis techniques going forward. We also introduce a new kind of circuit element called the "inductor"; Note 3B will be useful.

1. Coordinate Change of Basis: Examples

Consider a vector in the standard basis,

$$\vec{x} = a\vec{e}_1 + b\vec{e}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{I}\vec{x} \tag{1}$$

where, a, b are $\vec{x}$'s coordinates in the standard basis and $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the elementary standard basis vectors.

Given a new set of basis vectors, $\mathscr{V} = \{\vec{v}_1, \vec{v}_2\}$, if $\vec{x} \in \text{span}\{\mathscr{V}\}$, then we can find new coordinates in terms of this new basis. The new coordinates are called a_v, b_v and are described,

$$\vec{x} = a_v\vec{v}_1 + b_v\vec{v}_2 = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix} = \mathbf{V}\vec{x}_v \tag{2}$$

Now consider another set of basis vectors, $\mathscr{U} = \{\vec{u}_1, \vec{u}_2\}$. If $\vec{x} \in \text{span}\{\mathscr{U}\}$, then we can find the coordinates of $\vec{x}$ in terms of this basis. These coordinates are called a_u, b_u and are described,

$$\vec{x} = a_u\vec{u}_1 + b_u\vec{u}_2 = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \mathbf{U}\vec{x}_u \tag{3}$$

All of these bases are equivalent representations of any vector $\vec{x} \in \mathbb{R}^2$; each with their own set of coordinates. The same logic can, of course, be extended to any number of dimensions.

$$\vec{x} = \begin{bmatrix} | & | \\ \vec{e}_1 & \vec{e}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_u \\ b_u \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_v \\ b_v \end{bmatrix} \tag{4}$$

$$\vec{x} = I\vec{x} = \mathbf{V}\vec{x}_v = \mathbf{U}\vec{x}_u \tag{5}$$

Now that we've seen a conceptual overview of the change-of-basis, we can proceed with the worksheet problems.

(a) *Transformation From Standard Basis To Another Basis in* $\mathbb{R}^3$

Calculate the coordinate transformation between the following bases:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$ where $\vec{x}_u$ contains the coordinates of a vector in a basis of the columns of $\mathbf{U}$ and $\vec{x}_v$ is the coordinates of the same vector in the basis of the columns of $\mathbf{V}$.

Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_v$. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What is $\vec{x}_v$?

Answer: Note that $\vec{x}_u$ are coordinates in the standard basis. For any vector $\vec{x}$, we have that:

$$\vec{x} = \mathbf{U}\vec{x}_u = \mathbf{V}\vec{x}_v$$

Since $\mathbf{U} = \mathbf{I}$:

$$\vec{x}_v = \mathbf{V}^{-1}\mathbf{U}\vec{x}_u = \mathbf{V}^{-1}\vec{x}_u = \mathbf{T}\vec{x}_u$$

$$\mathbf{T} = \mathbf{V}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies \vec{x}_v = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \implies \vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \implies \vec{x}_v = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

(b) *Transformation Between Two Bases in* $\mathbb{R}^3$

Calculate the coordinate transformation between the following bases:

$$\mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{W} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_w = \mathbf{T}\vec{x}_v$. Let $\vec{x}_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and compute $\vec{x}_w$. Repeat this for $\vec{x}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Now let $\vec{x}_v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{x}_w$?

Answer: Again for any vector $\vec{x}$, we have that $\vec{x} = \mathbf{V}\vec{x}_v = \mathbf{W}\vec{x}_w$

$$\mathbf{T} = \mathbf{W}^{-1}\mathbf{V} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

2. Diagonalization

(a) Consider a matrix $\mathbf{A}$, a matrix $\mathbf{V}$ whose columns are the eigenvectors of $\mathbf{A}$, and a diagonal matrix $\mathbf{\Lambda}$ with the eigenvalues of $\mathbf{A}$ on the diagonal (in the same order as the eigenvectors (or columns) of $\mathbf{V}$). From these definitions, show that

$$\mathbf{AV} = \mathbf{V\Lambda}$$

Answer:

$$\mathbf{AV} = \mathbf{A}\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{A}\vec{v}_1 & \mathbf{A}\vec{v}_2 & \cdots & \mathbf{A}\vec{v}_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_k\vec{v}_k \\ | & | & & | \end{bmatrix}$$

$$\mathbf{V\Lambda} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_k\vec{v}_k \\ | & | & & | \end{bmatrix}$$

3. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage changes as a function of the derivative of the current across it. That is:

$$V_L(t) = L\frac{dI_L(t)}{dt}$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 1, we form the equivalent "fundamental" circuit for an inductor:

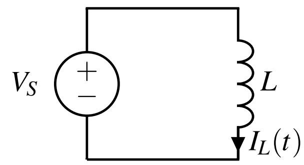

Figure 1: Inductor in series with a voltage source.

(a) What is the current through an inductor as a function of time? If the inductance is $L = 3\text{H}$, what is the current at $t = 6\text{s}$? Assume that the voltage source turns from 0V to 5V at time $t = 0\text{s}$, and there's no current flowing in the circuit before the voltage source turns on.

Answer: We proceed to analyze the given equation. Note that the voltage source is held at a constant value for $t \geq 0$, which allows us to express the derivative of current as a constant:

$$V_L(t) = L\frac{dI_L}{dt}$$
$$\frac{V_S}{L} = \frac{dI_L}{dt}$$

From here, we can see that the derivative of the current is a constant with respect to time! This immediately indicates that we have a linear relationship between current and time, with a slope set by the derivative. In terms of a general initial condition, the current is:

$$I_L(t) = \frac{V_S}{L}t + I_L(0)$$

So, the current in the inductor keeps growing over time! Inductors store energy in their magnetic field, so the more time that this voltage source feeds the inductor, the higher the current, and the greater the stored energy.

Substituting in the specific values asked for, $I_L(6\text{s}) = \frac{5\text{V}}{3\text{H}} \cdot 6\text{s} = 10\text{A}$.

(b) Now, we add some resistance in series with the inductor, as in Figure 2.
Solve for the current $I_L(t)$ in the circuit over time, in terms of R, L, V_S, t.

Answer: We begin by writing a KVL expression, and substititing in some known formulas for the inductor voltage and resistor voltage. There's also only a single current in the circuit (the one we're solving for, $I(t)$):

$$V_S = V_R(t) + V_L(t)$$
$$V_S = R\,I(t) + L\frac{d}{dt}I(t)$$
$$V_S - L\frac{d}{dt}I(t) = I(t) \cdot R$$
$$\frac{d}{dt}I(t) = -\frac{R}{L}I(t) + \frac{V_S}{L}$$

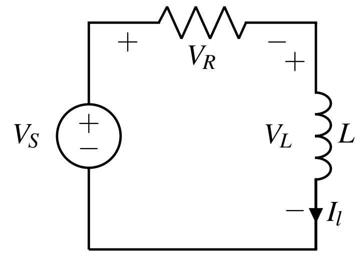

Figure 2: Inductor in series with a voltage source.

We recognize this as a first-order differential equation! With the practice we have had so far, we could jump straight to the solution for the current $I(t)$ since we know the initial current condition ($I(0) = 0$):

$$I(t) = \frac{V_S}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Alternatively, we can derive this solution using a change of variables, $\tilde{I}(t) = I(t) - \frac{V_S}{R}$, which leads us to:

$$\frac{d}{dt}\tilde{I}(t) = -\frac{R}{L}I(t)$$

Now, we find that $\tilde{I}(t) = ce^{-\frac{R}{L}t}$, with c to be solved for from the initial condition. Since $I(0) = 0$, and $I(t) = \tilde{I}(t) + \frac{V_S}{R} = ce^{-\frac{R}{L}t} + \frac{V_S}{R}$, we can say that $c = -\frac{V_S}{R}$, and so:

$$\begin{aligned} I(t) &= -\frac{V_S}{R}e^{-\frac{R}{L}t} + \frac{V_S}{R} \\ &= \frac{V_S}{R}\left(1 - e^{-\frac{R}{L}t}\right) \end{aligned}$$

This is the same current going through the resistor, so we can combine this information with the KVL to conclude that:

$$V_L(t) = V_S\left(e^{-\frac{R}{L}t}\right)$$

(c) **(Practice)** Suppose $R = 500\Omega, L = 1\text{mH}, V_S = 5\text{V}$. Plot the current through and voltage across the inductor ($I_L(t), V_L(t)$), as these quantities evolve over time.

Answer: The current begins at 0A and over time, the inductor begins to look like a short. In the long-term, the current settles to $\frac{V_S}{R}\text{A} = 1\text{mA}$. The voltage begins at $V_S = 5\text{V}$ because the inductor initially looks like an open circuit, and this voltage decreases exponentially over time down to zero.

The time constant governing both of these transient curves is $\tau = \frac{L}{R} = 2\mu\text{s}$. Using this information, we can sketch the curves for current (Figure 3) and inductor voltage (Figure 4). Notice that it is perfectly fine for the voltage to be discontinuous, but the same is not true for the current.

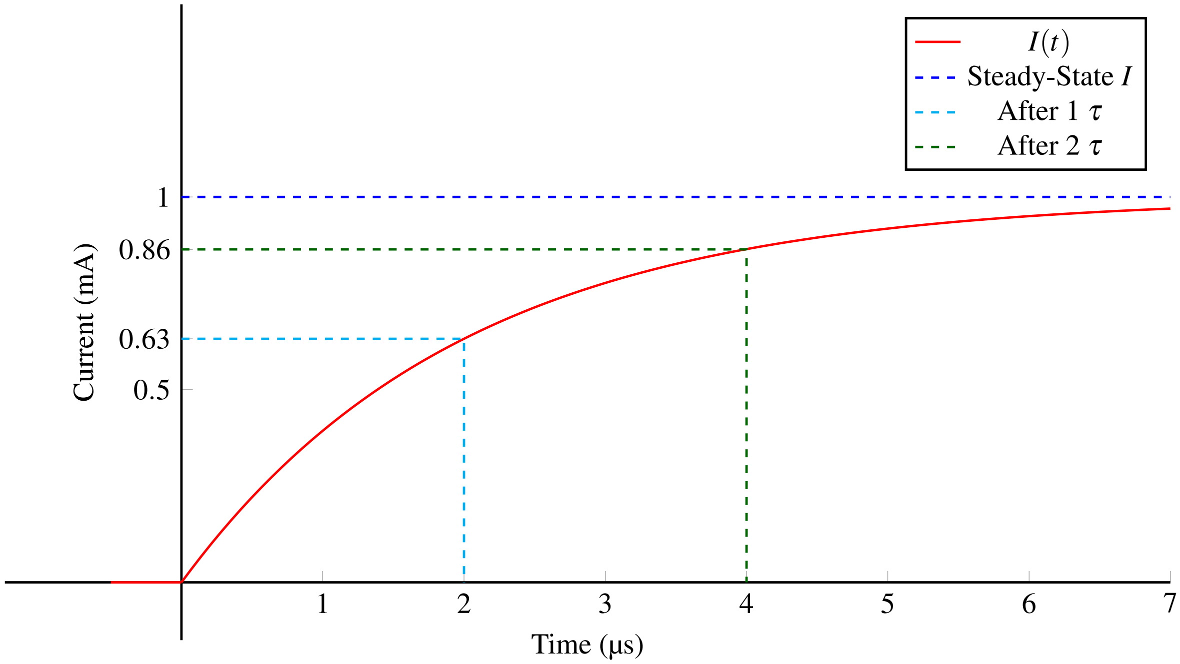

Figure 3: Transient Current in an RL circuit (with initial current $I(0) = 0$A.)

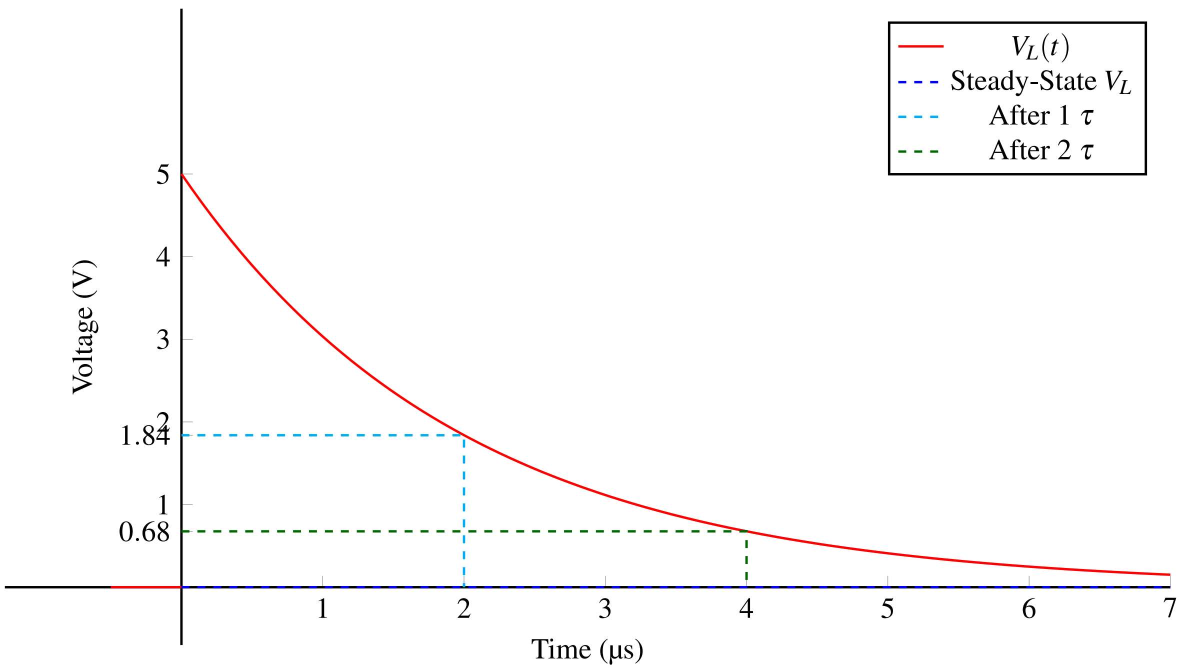

Figure 4: Transient Voltage across the inductor in an RL circuit (with initial current $I(0) = 0$A.)

4. Fibonacci Sequence

(a) The Fibonacci sequence is built as follows: the n-th number (F_n) is sum of the previous two numbers in the sequence. That is:

$$F_n = F_{n-1} + F_{n-2}$$

If the sequence is initialized with $F_1 = 0$ and $F_2 = 1$, then the first 11 numbers in the Fibonacci sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

We can express this computation as a matrix multiplication:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

What is **A**?

Answer: We can write the equations:

$$\begin{aligned} F_n &= 1 \cdot F_{n-1} + 1 \cdot F_{n-2} \\ F_{n-1} &= 1 \cdot F_{n-1} + 0 \cdot F_{n-2} \end{aligned}$$

From these, we conclude that

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

(b) Find the eigenvalues and corresponding eigenvectors of **A**.

Answer: We follow the familiar procedure from 16A, solving for λ where $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$. This gives us the characteristic polynomial of $\mathbf{A} - \lambda \mathbf{I}$:

$$\begin{aligned} (1-\lambda)(0-\lambda) - 1 \cdot 1 &= 0 \\ \lambda^2 - \lambda - 1 &= 0 \end{aligned}$$

The roots here are $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$. We now solve for the nullspaces of $\mathbf{A} - \lambda_1 \mathbf{I}$ and $\mathbf{A} - \lambda_2 \mathbf{I}$:

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}$$

By inspection here, $\vec{v}_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$. Similarly for λ_2:

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 1 - \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1-\sqrt{5}}{2} \end{bmatrix}$$

So $\vec{v}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$. In summary, the $\lambda, \vec{v}$ pairs are:

$$\left(\frac{1+\sqrt{5}}{2}, \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \right) \qquad \left(\frac{1-\sqrt{5}}{2}, \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \right)$$

(c) Diagonalize $\mathbf{A}$ (that is, in the expression $\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$, solve for each component matrix.)

Answer: Recall the 2×2 inverse formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Using these facts, first we form the eigenvectors matrix and its inverse:

$$\mathbf{V} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \qquad \mathbf{V}^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

Now, we can diagonalize $\mathbf{A}$:

$$\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \right)$$

(d) Use the diagonalized result to show that we can arrive at an analytical result for any F_n:

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n-1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n-1}$$

Answer: We have that F_n is equal to the first element of $\mathbf{A}^{n-2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Using $\mathbf{A}$'s diagonal form, we can see how raising it to some large power k reduces to a much simpler problem; raising a diagonal matrix to the same k-th power.

$$\mathbf{A}^k = \left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \right)^k$$

$$\mathbf{A}^k = \underbrace{\left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \right) \left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \right) \ldots \left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \right)}_{k \text{ times}}$$

$$\mathbf{A}^k = \underbrace{\left(\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \ldots \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \right)}_{k \text{ times}}$$

$$\mathbf{A}^k = \left(\mathbf{V} \underbrace{\boldsymbol{\Lambda}\boldsymbol{\Lambda} \ldots \boldsymbol{\Lambda}}_{k \text{ times}} \mathbf{V}^{-1} \right)$$

$$\mathbf{A}^k = \mathbf{V}\boldsymbol{\Lambda}^k\mathbf{V}^{-1}$$

Now, we can proceed to simplify the expression for F_n:

$$
\begin{aligned}
F_n &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{n-2} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n-1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n-1}
\end{aligned}
$$

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