EECS 16B Designing Information Devices and Systems II
Spring 2021 Discussion Worksheet

Discussion 4A

The relevant notes for this discussion is Note 3A.

1. Changing Coordinates and Systems of Differential Equations, II

In the previous discussion we analyzed and solved a pair of differential equations where the variables of interest were coupled.

$$\frac{d}{dt}z_1(t) = -5z_1(t) + 2z_2(t)$$
$$\frac{d}{dt}z_2(t) = 6z_1(t) - 6z_2(t).$$

We solved this system by using a coordinate transformation that gave us a decoupled system of equations. In the last discussion we were simply handed this transformation, but in this discussion we will construct the transformation for ourselves.

We will focus our explorations on the voltages across the capacitors in the following circuit.

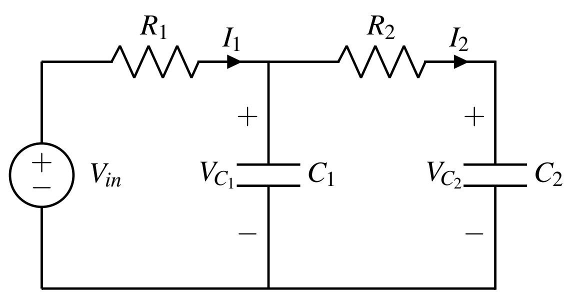

Figure 1: Two dimensional system: a circuit with two capacitors, like the one in lecture.

(a) Write the system of differential equations governing the voltages across the capacitors V_{C_1}, V_{C_2}. Use the following values: $C_1 = 1\mu F, C_2 = \frac{1}{3}\mu F, R_1 = \frac{1}{3}M\Omega, R_2 = \frac{1}{2}M\Omega$.

Answer: Start by solving for the currents and voltages across the capacitors:

$$V_{C_2} = V_{C_1} - I_2R_2, \quad I_2 = C_2\frac{d}{dt}V_{C_2}$$

$$I_1 = I_2 + C_1\frac{d}{dt}V_{C_1}, \quad V_{in} - I_1R_1 = V_1$$

Yields,

$$I_1 = \frac{V_{in}}{R_1} - \frac{V_{C_1}}{R_1}, \quad I_2 = \frac{V_{C_1}}{R_2} - \frac{V_{C_2}}{R_2}$$

Now, we can plug into the formula for current across a capacitor:

$$\begin{aligned}
\frac{d}{dt}V_{C_1} &= \frac{1}{C_1}(I_1 - I_2) \\
&= \frac{1}{C_1}\left(\frac{V_{in}}{R_1} - \frac{V_{C_1}}{R_1} - \frac{V_{C_1}}{R_1} + \frac{V_{C_2}}{R_2}\right) \\
&= -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)V_{C_1} + \frac{V_{C_2}}{R_2C_1} + \frac{V_{in}}{R_1C_1} \\
\frac{d}{dt}V_{C_2} &= \frac{1}{C_2}(I_2) \\
&= \frac{V_{C_1}}{R_2C_2} - \frac{V_{C_2}}{R_2C_2}
\end{aligned}$$

Now group the terms into a matrix with the values given above,

$$\begin{bmatrix} \frac{d}{dt}V_{C_1}(t) \\ \frac{d}{dt}V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_{in}(t)$$

Plugging in the values for R, C yields:

$$\begin{bmatrix} \frac{d}{dt}V_{C_1}(t) \\ \frac{d}{dt}V_{C_2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}(t)$$

(b) Suppose also that V_{in} was at 7 Volts for a long time, and then transitioned to be 0 Volts at time $t = 0$. This "new" system of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}y_1(t) = -5y_1(t) + 2y_2(t) \tag{1}$$

$$\frac{d}{dt}y_2(t) = 6y_1(t) - 6y_2(t) \tag{2}$$

with initial conditions $y_1(0) = 7$ and $y_2(0) = 7$.
Write out the differential equations and initial conditions in matrix/vector form.

Answer:

$$\begin{bmatrix} \frac{d}{dt}y_1(t) \\ \frac{d}{dt}y_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

We will define the differential matrix as A_y, where

$$A_y = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$$

(c) Find the eigenvalues $\lambda_1,\ \lambda_2$ and eigenspaces for the matrix corresponding to the differential equation matrix above.

Answer: Eigenvalues λ and eigenvectors v of matrix A are given by

$$A_y v = \lambda v.$$

In order to find the eigenvalues, we take the determinant:

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} -5-\lambda & 2 \\ 6 & -6-\lambda \end{bmatrix}\right) = 0$$

We can solve this using a 2×2 determinant form seen in 16A,

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc,$$

or by Gaussian elimination.

$$\begin{aligned} (-5-\lambda)(-6-\lambda) - 12 &= 0 \\ 30 + 11\lambda + \lambda^2 - 12 &= 0 \\ \lambda^2 + 11\lambda + 18 &= 0 \\ (\lambda+9)(\lambda+2) &= 0 \end{aligned}$$

Giving:

$$\lambda = -9, -2$$

The eigenspace associated with $\lambda_1 = -9$ is given by:

$$\begin{bmatrix} -5+9 & 2 \\ 6 & -6+9 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix} \vec{v}_{\lambda_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\vec{v}_{\lambda_1} = \alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The eigenspace associated with $\lambda_2 = -2$ is given by:

$$\begin{bmatrix} -5+2 & 2 \\ 6 & -6+2 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix} \vec{v}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\vec{v}_{\lambda_2} = \beta \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(d) Change coordinates into the eigenbasis to re-express the differential equations in terms of new variables $z_{\lambda_1}(t),\ z_{\lambda_2}(t)$. (These variables represent eigenbasis-aligned coordinates.)

Answer:

$$\vec{y} = \vec{v}_{\lambda_1} z_{\lambda_1} + \vec{v}_{\lambda_2} z_{\lambda_2}$$
$$\vec{y} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_{\lambda_1} \\ z_{\lambda_2} \end{bmatrix}$$

We can define the change-of-coordinates matrix from the eigenbasis to our original basis as:

$$V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \implies V^{-1} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

Changing coordinates to the eigenbasis:

$$\begin{bmatrix} z_{\lambda_1} \\ z_{\lambda_2} \end{bmatrix} = V^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
$$A_{z_\lambda} = V^{-1} A_y V = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 9 & -4 \\ -18 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}$$

That is:

$$\begin{bmatrix} \frac{d}{dt} z_{\lambda_1}(t) \\ \frac{d}{dt} z_{\lambda_2}(t) \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_{\lambda_1}(t) \\ z_{\lambda_2}(t) \end{bmatrix}$$

(e) Solve the differential equation for $z_{\lambda_i}(t)$ in the eigenbasis.

Answer: First we get the initial condition:

$$\vec{z}_\lambda(0) = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Then we solve based on the form of the problem and our previous differential equation experience:

$$\vec{z}_\lambda(t) = \begin{bmatrix} K_1 e^{-9t} \\ K_2 e^{-2t} \end{bmatrix}$$

Plugging in for the initial condition gives:

$$\vec{z}_\lambda(t) = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

(f) Convert your solution back into the original coordinates to find $y_i(t)$.

Answer:

$$\vec{y}(t) = V\vec{z}_\lambda(t) = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix}$$

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