EECS 16B Spring 2021 — Designing Information Devices and Systems II Discussion Worksheet

Discussion 3B

The relevant notes for this discussion are the sections 1 and 2 of Note 3.

1. Changing Coordinates and Systems of Differential Equations

Suppose we have the pair of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}x_1(t) = -9x_1(t)$$

$$\frac{d}{dt}x_2(t) = -2x_2(t)$$

with initial conditions $x_1(0) = -1$ and $x_2(0) = 3$.

(a) Solve for $x_1(t)$ and $x_2(t)$ for $t \geq 0$.

Answer: From experience, we know the solution to these differential equations are of the form:

$$\begin{aligned} x_1(t) &= K_1 e^{-9t} \\ x_2(t) &= K_2 e^{-2t} \end{aligned} \tag{1}$$

Plugging in for initial conditions:

$$\begin{aligned} x_1(0) &= K_1 e^0 = -1 \\ K_1 &= -1 \\ x_2(0) &= K_2 e^0 = 3 \\ K_2 &= 3 \end{aligned} \tag{2}$$

Giving:

$$\begin{aligned} x_1(t) &= -e^{-9t} \\ x_2(t) &= 3e^{-2t} \end{aligned} \tag{3}$$

Suppose we are actually interested in a different set of variables with the following differential equations:

$$\begin{aligned} \frac{d}{dt}z_1(t) &= -5z_1(t) + 2z_2(t) \\ \frac{d}{dt}z_2(t) &= 6z_1(t) - 6z_2(t). \end{aligned}$$

(b) Write out the above system of differential equations in matrix form. Assuming that the initial state $z(\vec{0}) = \begin{bmatrix} 7 & 7 \end{bmatrix}^T$, can we solve this system directly?

Answer:

$$\frac{d}{dt}\left(\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}\right) = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \cdot \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix}$$

Let's call the transition matrix A for convenience.

$$A = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$$

We cannot solve this system directly.

(c) Consider that in our frustration with the previous system of differential equations, we start hearing voices. These voices whisper to us that that we should try the following change of variables:

$$z_1(t) = -y_1(t) + 2y_2(t)$$
$$z_2(t) = 2y_1(t) + 3y_2(t).$$

Write out this transformation in matrix form ($\vec{z} = V\vec{y}$).

Answer:

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

We can call the forward change-of-coordinates matrix V:

$$V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$

To transform back from $\vec{z}$ to $\vec{y}$, we can use the inverse of V.

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

For each of the parts (d) - (f), solve the questions two ways: 1. using direct substitution, and 2. using matricies and vectors .

(d) How do the initial conditions for $z_i(t)$ translate into the initial conditions for $y_i(t)$?

1. By direct substitution:

Answer: Plugging in the known initial values of $z_i(t)$ yields the following system of equations:

$$z_1(0) = 7 = -y_1(0) + 2y_2(0)$$
$$z_2(0) = 7 = 2y_1(0) + 3y_2(0)$$

Solving this system gives:

$$y_1(0) = -1$$
$$y_2(0) = 3$$

2. Using matrices and vectors:

Answer: We can transform from $\vec{z}$ back to $\vec{y}$ using V^{-1}.

$$V^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{7}\begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(e) Rewrite the differential equations in terms of $y_i(t)$. Can we solve this system of differential equations?

1. By direct substitution:

Answer: We can solve our system of equations relating $z_i(t)$ and $y_i(t)$ for $y_i(t)$:

$$y_1(t) = \frac{-3}{7}z_1(t) + \frac{2}{7}z_2(t)$$
$$y_2(t) = \frac{2}{7}z_1(t) + \frac{1}{7}z_2(t).$$

Taking the derivative of these two equations gives:

$$\frac{dy_1(t)}{dt} = \frac{-3}{7} \cdot \frac{dz_1(t)}{dt} + \frac{2}{7} \cdot \frac{dz_2(t)}{dt}$$
$$\frac{dy_2(t)}{dt} = \frac{2}{7} \cdot \frac{dz_1(t)}{dt} + \frac{1}{7} \cdot \frac{dz_2(t)}{dt}.$$

Plugging in the values of $\frac{d}{dt}z_i(t)$ from the original differential equations gives:

$$\begin{aligned}\frac{dy_1(t)}{dt} &= \frac{-3}{7} \cdot (-5z_1(t) + 2z_2(t)) + \frac{2}{7} \cdot (6z_1(t) - 6z_2(t)) \\ &= \frac{27}{7}z_1(t) + \frac{-18}{7}z_2(t) \\ \frac{dy_2(t)}{dt} &= \frac{2}{7} \cdot (-5z_1(t) + 2z_2(t)) + \frac{1}{7} \cdot (6z_1(t) - 6z_2(t)) \\ &= \frac{-4}{7}z_1(t) + \frac{-2}{7}z_2(t).\end{aligned}$$

Simplifying the right hand sides and rewriting in terms of $y_i(t)$ gives the differential equations we seek:

$$\frac{dy_1(t)}{dt} = -9y_1(t) \tag{4}$$
$$\frac{dy_2(t)}{dt} = -2y_2(t). \tag{5}$$

Since these equations are decoupled, we can easily solve them. In fact, these equations and initial conditions are identical to the $x_i(t)$ differential equations we solved in the part (a).

2. Using matrices and vectors:

Answer: At this point we have the following equations:

$$\frac{d\vec{z}}{dt} = A\vec{z} \tag{6}$$

$$\vec{z} = V\vec{y} \tag{7}$$

We want to find the matrix B such that:

$$\frac{d\vec{y}}{dt} = B\vec{y} \tag{8}$$

By plugging eq. (7) into eq. (8), and expanding the result

$$\begin{aligned}\frac{d\vec{y}}{dt} &= \frac{d}{dt}(V^{-1}\vec{z}) \\ &= V^{-1}\frac{d}{dt}(\vec{z}) \\ &= V^{-1}A\vec{z} \\ &= V^{-1}AV\vec{y}\end{aligned}$$

we find that $B = V^{-1}AV$ and get our differential equation for $\vec{y}$.

$$\begin{aligned}\begin{bmatrix}\frac{dy_1(t)}{dt} \\ \frac{dy_2(t)}{dt}\end{bmatrix} &= \begin{bmatrix}-\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7}\end{bmatrix}\begin{bmatrix}-5 & 2 \\ 6 & -6\end{bmatrix}\begin{bmatrix}-1 & 2 \\ 2 & 3\end{bmatrix}\begin{bmatrix}y_1(t) \\ y_2(t)\end{bmatrix} \\ &= \begin{bmatrix}-9 & 0 \\ 0 & -2\end{bmatrix}\begin{bmatrix}y_1(t) \\ y_2(t)\end{bmatrix}\end{aligned}$$

(f) What are the solutions for $z_i(t)$?

1. Solve this with direct substitution:

Answer: Recognize that the differential equations and initial conditions are the same for $\vec{x}$ and $\vec{y}$. Thus we know that

$$\begin{aligned}y_1(t) &= x_1(t) = -e^{-9t} \\ y_2(t) &= x_2(t) = 3e^{-2t}\end{aligned}$$

Plugging this into the equation for $\vec{z}$ gives:

$$\begin{aligned}z_1(t) &= -y_1(t) + 2y_2(t) = e^{-9t} + 6e^{-2t} \\ z_2(t) &= 2y_1(t) + 3y_2(t) = -2e^{-9t} + 9e^{-2t}\end{aligned}$$

2. Solve this with matrices and vectors:

Answer: The solution for $z_i(t)$ is:

$$\begin{bmatrix}z_1(t) \\ z_2(t)\end{bmatrix} = \begin{bmatrix}-1 & 2 \\ 2 & 3\end{bmatrix}\begin{bmatrix}y_1(t) \\ y_2(t)\end{bmatrix} = \begin{bmatrix}-1 & 2 \\ 2 & 3\end{bmatrix}\begin{bmatrix}-e^{-9t} \\ 3e^{-2t}\end{bmatrix} = \begin{bmatrix}e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t}\end{bmatrix}$$

You can verify that the initial conditions for $z_i(t)$ and differential equation for $\frac{d}{dt}z_i(t)$ hold by plugging in for these solutions of $z_1(t)$ and $z_2(t)$.

Contributors:

- Anant Sahai.
- Regina Eckert.
- Nathan Lambert.
- Kareem Ahmad.