EECS 16B Spring 2021 — Designing Information Devices and Systems II — Discussion Worksheet

Discussion 3A

For this discussion, Note 2 is helpful.

1. Differential equations with piecewise constant inputs

Working through this question will help you understand better differential equations with inputs. Along the way, we will also touch a bit on going from continuous-time into a discrete-time view. This problem also provides a vehicle to review relevant concepts from calculus.

(a) Consider the scalar system

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t). \tag{1}$$

Our goal is to solve this system (find an appropriate function $x(t)$) for general inputs $u(t)$. To do this, we will start with a piecewise constant $u(t)$; we already have the tools to solve this system, which we will do in the first few parts. Later in the worksheet, we will extend this to general $u(t)$.

Suppose that $x(t)$ is continuous (in real systems, this is almost always true). Further suppose that our input $u(t)$ of interest is piecewise constant over durations of width Δ. In other words:

$$u(t) = u(i\Delta) = u[i] \text{ if } t \in [i\Delta, (i+1)\Delta). \tag{2}$$

In keeping with this notation, we will use the notation

$$x_d[i] = x(i\Delta).$$

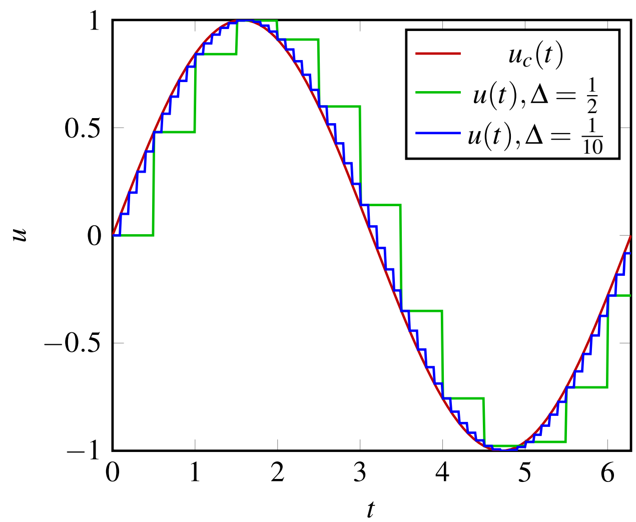

Figure 1: An example of a discrete input where the limit as the time-step Δ goes to 0 approaches a continuous function. The red line, the original signal $u_c(t) = \sin(t)$, is traced almost exactly by the blue line, which has a small time-step, and not nearly as well by the green line, which has a large time-step.

The first step to analyzing this system is to discover its behavior across a time-step with constant input, since we already know how to solve these kinds of systems.
Given that we know the value of $x(i\Delta) = x_d[i]$, compute $x_d[i+1] = x((i+1)\Delta)$.
Hint: For $t \in [i\Delta, (i+1)\Delta)$, the system is

$$\frac{d}{dt}x(t) = \lambda x(t) + u[i].$$

Also see Note 2.
Here is a solution to this system, which may help with visual intuition:

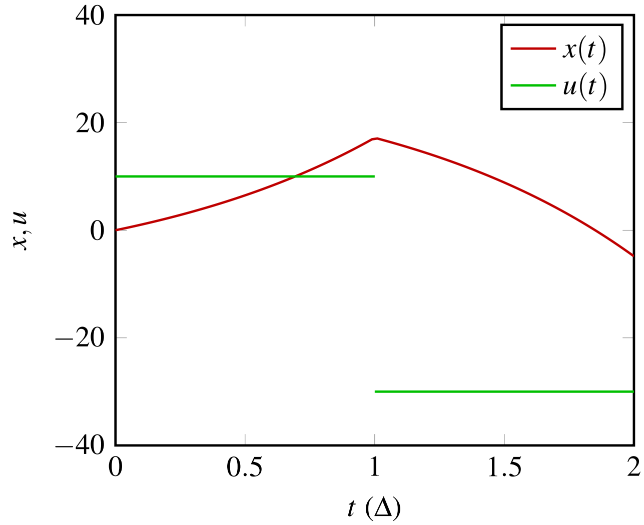

Figure 2: An example of a solution to this diff. eq. system. In this case $\lambda = 1, u[0] = 10, u[1] = -30$.

Answer:
If $t \in [i\Delta, (i+1)\Delta)$, the differential equation takes the form

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) = \lambda x(t) + u[i]. \tag{3}$$

From Note 2 we know that the solution ought to have the form

$$x(t) = \alpha e^{\lambda(t-i\Delta)} + \beta$$

Why is it in terms of $t - i\Delta$? We're given the value $x_d[i] = x(i\Delta)$, and we really want to model the growth of x between $i\Delta$ and t. Intuitively, this should be independent of the values of $i\Delta$ and t and only dependent on their difference.
Now we try to fit $x(t)$ to the diff. eq. (3), which we do first, as well as the initial condition $x(i\Delta) = x_d[i]$.
To fit $x(t)$ to eq. (3), we equate the LHS of eq. (3) to the RHS. The LHS is

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left(\alpha e^{\lambda(t-i\Delta)} + \beta\right) = \lambda \alpha e^{\lambda(t-i\Delta)}$$

so equating the LHS with the RHS gives

$$\begin{aligned}\lambda\alpha e^{\lambda(t-i\Delta)} &= \lambda x(t) + u[i] = \lambda\left(\alpha e^{\lambda(t-i\Delta)} + \beta\right) + u[i] \\ &= \lambda\alpha e^{\lambda(t-i\Delta)} + \lambda\beta + u[i] \\ \implies 0 &= \lambda\beta + u[i] \\ \implies \beta &= -\frac{u[i]}{\lambda}.\end{aligned}$$

Now we want to use the initial condition $x(i\Delta) = x_d[i]$. Expanding $x(i\Delta)$ as per our guess,

$$x_d[i] = x(i\Delta) = \alpha e^{\lambda(i\Delta - i\Delta)} + \beta = \alpha + \beta$$

And using $\beta = -\frac{u[i]}{\lambda}$ we get

$$\begin{aligned}x_d[i] &= \alpha + \frac{-u[i]}{\lambda} \\ \implies \alpha &= x_d[i] + \frac{u[i]}{\lambda}.\end{aligned}$$

Now we have the values of both α and β, which is all we need to write $x(t)$ fully. So for $t \in [i\Delta, (i+1)\Delta)$ (which is the assumption we made for eq. (3) to hold),

$$\begin{aligned}x(t) = \alpha e^{\lambda(t-i\Delta)} + \beta &= \left(x_d[i] + \frac{u[i]}{\lambda}\right) e^{\lambda(t-i\Delta)} - \frac{u[i]}{\lambda} \\ &= e^{\lambda(t-i\Delta)} x_d[i] + \frac{e^{\lambda(t-i\Delta)} - 1}{\lambda} u[i]\end{aligned}$$

The reason we simplify in this manner is because we want to split the value of $x(t)$ into the effect of the initial condition $x_d[i]$, and the input $u[i]$. Now we can see how each independent part affects $x(t)$.

Now since $x(t)$ is continuous across all t, $x_d[i+1] = x((i+1)\Delta)$. This may seem obvious; the continuity condition just ensures that the function doesn't have bad behavior at only the points $i\Delta$ or $(i+1)\Delta$. Of course, these discontinuities don't happen in real systems, so our assumption makes sense. Thus

$$\begin{aligned}x_d[i+1] = x((i+1)\Delta) &= e^{\lambda((i+1)\Delta - i\Delta)} x_d[i] + \frac{e^{\lambda((i+1)\Delta - i\Delta)} - 1}{\lambda} u[i] \\ &= e^{\lambda\Delta} x_d[i] + \frac{e^{\lambda\Delta} - 1}{\lambda} u[i].\end{aligned}$$

This is the quantity we want.

(b) Now that we've found a one-step recurrence for $x_d[i+1]$ in terms of $x_d[i]$, we want to get an expression for $x_d[i]$ in terms of the original value $x(0) = x_d[0]$, and all the inputs u. This is so that we can eventually convert this function for $x_d[i]$ into a function for $x(t)$.

Unroll the implicit recursion you derived in the previous part to write $x_d[i+1]$ as a sum that involves $x_d[0]$ and the $u[j]$ for $j = 0, 1, \ldots, i$.

For this part, feel free to just consider the discrete-time system in a simpler form

$$x_d[i+1] = ax_d[i] + bu[i] \tag{4}$$

and you don't need to worry about what a and b actually are in terms of λ and Δ.
(*Hint*: What is $x_d[1]$ in terms of $x_d[0]$? What is $x_d[2]$ in terms of (only) $x_d[0]$? What about $x_d[3]$? Can you find a pattern?)

Answer: Let's look at the pattern, given that

$$x_d[i+1] = ax_d[i] + bu[i].$$

Starting from $i = 0$, we get

$$\begin{aligned}
x_d[1] &= ax_d[0] + bu[0] \\
x_d[2] &= ax_d[1] + bu[1] = a\left(ax_d[0] + bu[0]\right) + bu[1] \\
&= a^2 x_d[0] + b\left(au[0] + u[1]\right) \\
x_d[3] &= ax_d[2] + bu[2] = a\left(a^2 x_d[0] + b\left(au[0] + u[1]\right)\right) + bu[2] \\
&= a^3 x_d[0] + b\left(u[2] + au[1] + a^2 u[0]\right)
\end{aligned}$$

The idea is to collect terms with all the x_d's in one term and all the u's in the other term. Again, this separates out the effect of the initial condition $x_d[0]$ and all the inputs $u[j]$.
So, given this pattern, we guess

$$x_d[i] = a^i x_d[0] + b\sum_{j=0}^{i-1} a^{i-1-j} u[j]. \tag{5}$$

Let's check that this works. The way we do this is compute $x_d[i+1]$ through this formula, and also from eq. (4), and check that they're equal.

$$\begin{aligned}
x_d[i+1] = ax_d[i] + bu[i] &= a\left(a^i x_d[0] + b\sum_{j=0}^{i-1} a^{i-1-j} u[j]\right) + bu[i] \\
&= a^{i+1} x_d[0] + b\left(\sum_{j=0}^{i-1} a^{i-j} u[j]\right) + bu[i] \\
&= a^{i+1} x_d[0] + b\left(u[i] + \sum_{j=0}^{i-1} a^{i-j} u[j]\right) \\
&= a^{i+1} x_d[0] + b\sum_{j=0}^{i} a^{i-j} u[j]
\end{aligned}$$

This satisfies eq. (5), for $i+1$ and hence our guess was correct!

(c) For a given time t in continuous real time, what is the discrete i interval that corresponds to it?
(*Hint*: $\lfloor x \rfloor$ is the largest integer smaller than x.)

Answer:

$$i = \left\lfloor \frac{t}{\Delta} \right\rfloor$$

is the discrete time index i that corresponds to the time t in real time, because it is the only i satisfying $t \in [i\Delta, (i+1)\Delta)$.

(d) Here's the first payoff! Use the results of part (a) and (b) to give an approximate expression for $x(t)$ for any t, in terms of $x_d[0] = x(0)$ and the inputs $u[j]$. You can assume that Δ is small enough that $x(t)$ does not change too much (is approximately constant) over an interval of length Δ.

(*Hint*: The assumption we just made allows us to approximate $x(t) \approx x\left(\Delta\lfloor \frac{t}{\Delta} \rfloor\right) = x_d\left[\lfloor \frac{t}{\Delta} \rfloor\right]$.)

Answer: Using the result derived in part (c) and the assumption,

$$x(t) \approx x\left(\Delta \left\lfloor \frac{t}{\Delta} \right\rfloor\right) = x_d\left[\left\lfloor \frac{t}{\Delta} \right\rfloor\right].$$

Using the result from part (b),

$$x(t) \approx a^{\lfloor \frac{t}{\Delta} \rfloor} x_d[0] + b \sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} a^{\lfloor \frac{t}{\Delta} \rfloor - 1 - j} u[j]$$

It remains to find a and b. Fitting the result of part (a) to (4),

$$a = e^{\lambda\Delta} \qquad \text{and} \qquad b = \frac{e^{\lambda\Delta} - 1}{\lambda}.$$

Thus

$$x(t) \approx \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_d[0] + \frac{e^{\lambda\Delta} - 1}{\lambda} \sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} (e^{\lambda\Delta})^{\lfloor \frac{t}{\Delta} \rfloor - 1 - j} u[j].$$

We observe that the initial condition $x_d[0]$ has an exponential (in t) effect on $x(t)$, and inputs at the beginning have exponential (again in t) effect on $x(t)$, with the later inputs having an exponentially decaying effect on $x(t)$ relative to the earlier inputs. (It's exponentials all the way down.)

(e) Now, we are going to turn this around. Suppose that the $u[i]$ is actually a sample of a desired input $u_c(t)$ in continuous time. Namely, suppose that $u[i] = u_c(i\Delta)$.

To clarify, $u(t)$ is a piecewise constant function; $u[i]$ is the discrete input that constructs $u(t)$; and $u_c(t)$ is the underlying input $u[i]$ is sampled from.

The underlying goal is to find an expression for $x(t)$ in the limit $\Delta \to 0$, in terms of $u_c(t)$ and the initial condition $x(0)$. To this end, start by substituting an appropriate value of u_c for u in the result from part (d). (Note: don't take any limits in this problem; just do the substitution.)

Answer: Using the substitution $u_c(j\Delta)$ for $u[j]$ in part (e), we get

$$\begin{aligned} x(t) &\approx \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_d[0] + \frac{e^{\lambda\Delta} - 1}{\lambda} \sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} (e^{\lambda\Delta})^{\lfloor \frac{t}{\Delta} \rfloor - 1 - j} u[j] \\ &= \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_d[0] + \frac{e^{\lambda\Delta} - 1}{\lambda} \sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} (e^{\lambda\Delta})^{\lfloor \frac{t}{\Delta} \rfloor - 1 - j} u_c(j\Delta). \end{aligned}$$

(f) We want to take the limit $\Delta \to 0$ of our (discrete-time) expression and thus get a continuous-time function, but right now our discrete-time expression itself is pretty complicated. Let's simplify it by making some approximations which become exact in the limit.

Further approximate the previous expression by considering the following two estimates:

i. Let $n = \lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$ where needed and treat $\Delta \approx \frac{t}{n}$. This is a meaningful approximation when we think about n large enough.

ii. Treat $\frac{1-e^{-\lambda\Delta}}{\lambda} \approx \Delta$. This is a meaningful approximation when we think about Δ small enough. One can derive this estimate by using Taylor's theorem from calculus, but it's not required here.

(*Hint*: Use the first estimate to get rid of "floor" terms, then use both estimates to simplify further.)

Answer: The first estimate justifies getting rid of the "floor" terms. We have a lot of those terms, so it's good to use it here.

Plugging in $n = \lfloor \frac{t}{\Delta} \rfloor \approx \frac{t}{\Delta}$ into the result of part (e),

$$\begin{aligned}
x(t) &\approx \left(e^{\lambda\Delta}\right)^{\lfloor \frac{t}{\Delta} \rfloor} x_d[0] + \frac{e^{\lambda\Delta}-1}{\lambda} \sum_{j=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} (e^{\lambda\Delta})^{\lfloor \frac{t}{\Delta} \rfloor - 1 - j} u_c(j\Delta) \\
&\approx \left(e^{\lambda\Delta}\right)^{\frac{t}{\Delta}} x_d[0] + \frac{e^{\lambda\Delta}-1}{\lambda} \sum_{j=0}^{\frac{t}{\Delta} - 1} (e^{\lambda\Delta})^{\frac{t}{\Delta} - 1 - j} u_c(j\Delta) \\
&\approx e^{\lambda t} x_d[0] + \frac{e^{\lambda\Delta}-1}{\lambda} \sum_{j=0}^{n-1} e^{\lambda t - \lambda\Delta - \lambda\Delta j} u_c(j\Delta) \\
&= e^{\lambda t} x_d[0] + e^{\lambda t - \lambda\Delta} \frac{e^{\lambda\Delta}-1}{\lambda} \sum_{j=0}^{n-1} e^{-\lambda\Delta j} u_c(j\Delta) \\
&= e^{\lambda t} x_d[0] + e^{\lambda t} \frac{1-e^{-\lambda\Delta}}{\lambda} \sum_{j=0}^{n-1} e^{-\lambda\Delta j} u_c(j\Delta)
\end{aligned}$$

The next thing to do is to use the second given estimate. The term in the second given estimate is exactly one of the terms in our main approximation, so simply applying the estimate gives

$$\begin{aligned}
x(t) &\approx e^{\lambda t} x_d[0] + e^{\lambda t} \frac{1-e^{-\lambda\Delta}}{\lambda} \sum_{j=0}^{n-1} e^{-\lambda\Delta j} u_c(j\Delta) \\
&\approx e^{\lambda t} x_d[0] + e^{\lambda t} \Delta \sum_{j=0}^{n-1} e^{-\lambda\Delta j} u_c(j\Delta).
\end{aligned}$$

The last thing to do is to use the estimate $\Delta \approx \frac{t}{n}$, so as to avoid having all three of t, n, Δ in the same expression, when one of those quantities is redundant.

$$\begin{aligned}
x(t) &\approx e^{\lambda t} x_d[0] + e^{\lambda t} \Delta \sum_{j=0}^{n-1} e^{-\lambda\Delta j} u_c(j\Delta) \\
&\approx e^{\lambda t} x_d[0] + e^{\lambda t} \frac{t}{n} \sum_{j=0}^{n-1} e^{-\lambda j \frac{t}{n}} u_c\left(j\frac{t}{n}\right).
\end{aligned}$$

Note that the dependence of $x(t)$ on both $x_d[0]$ and the input u_c is the same; it's been preserved, and perhaps made more clear, through our approximations.

This may seem like a long solution, but the main idea is to just use the estimates one by one, and simplify as much as possible.

(g) Here's our second payoff! We now obtain a continuous-time expression for $x(t)$, completing the transition into continuous-time. Take the limit of $x(t)$ as $\Delta \to 0$ or equivalently as $n \to \infty$. What is the expression you get for $x(t)$?

(*Hint*: Remember your definition of definite integrals as limits of Riemann sums in calculus.)

Answer: Our groundwork from part (f) makes this easier. In particular, we take the limit as prescribed, and get

$$\begin{aligned}\lim_{\Delta\to 0} x(t) = \lim_{\frac{t}{n}\to 0} x(t) &= \lim_{n\to\infty} x(t) \\ &= \lim_{n\to\infty}\left(e^{\lambda t}x_d[0] + e^{\lambda t}\frac{t}{n}\sum_{j=0}^{n-1} e^{-\lambda j\frac{t}{n}} u_c\left(j\frac{t}{n}\right)\right) \\ &= e^{\lambda t}x_d[0] + e^{\lambda t}\lim_{n\to\infty}\sum_{j=0}^{n-1} e^{-\lambda j\frac{t}{n}} u_c\left(j\frac{t}{n}\right)\frac{t}{n}.\end{aligned}$$

Recall that the definite integral is defined from Riemann sums as

$$\int_0^t f(\tau)d\tau = \lim_{n\to\infty}\sum_{j=0}^{n-1} f(\tau_j^*)\Delta_j$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_n = t$, $\tau_j^* \in [\tau_{j-1}, \tau_j]$, and $\Delta_j = \tau_j - \tau_{j-1}$. This looks a lot like our limit, where

$$\tau_j = \tau_j^* = j\frac{t}{n} \qquad \text{and} \qquad \Delta_j = \frac{t}{n} \qquad \text{and} \qquad f(\tau) = e^{-\lambda\tau}u_c(\tau).$$

So we can convert the sum into an integral.

$$\begin{aligned}\lim_{\Delta\to 0} x(t) &= e^{\lambda t}x_d[0] + e^{\lambda t}\lim_{n\to\infty}\sum_{j=0}^{n-1} e^{-\lambda j\frac{t}{n}} u_c\left(j\frac{t}{n}\right)\frac{t}{n}. \\ &= e^{\lambda t}x_d[0] + e^{\lambda t}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau\end{aligned}$$

which is our final answer. We can't simplify further because we don't know the form of $u_c(\tau)$.

Note that the dependence of $x(t)$ on both $x_d[0]$ and the input u_c is the same. This is a special case of a crucial point: *sums of small quantities behave roughly the same as integrals*. This is one of the main ways to fluently transfer between discrete and continuous time.

(h) Verify the analytic solution for $x(t)$ found in (g) for $u(t) = 0$ and for $u(t) = u_0$.

Answer:

When $u(t) = 0$, the differential equation becomes

$$\frac{d}{dt}x(t) = \lambda x(t)$$

with initial condition $x(0) = x_d[0]$.

We know from lecture and notes that this has a solution of $x(t) = x_d[0]e^{\lambda t}$.

Plugging $u(t) = 0$ into the analytic solution:

$$\begin{aligned} x(t) &= e^{\lambda t} x_d[0] + e^{\lambda t} \int_0^t e^{-\lambda \tau} u_c(\tau) d\tau = e^{\lambda t} x_d[0] + e^{\lambda t} \int_0^t e^{-\lambda \tau} \cdot 0 d\tau \\ &= e^{\lambda t} x_d[0]. \end{aligned}$$

We see the two solutions are equal, so the formula holds.
When $u(t) = u_0$, we know how to solve the differential equation:

$$\frac{d}{dt} x(t) = \lambda x(t) + u_0$$

with initial condition $x(0) = x_d[0]$.
We can start by factoring the equation slightly:

$$\frac{d}{dt} x(t) = \lambda \left(x(t) + \frac{u_0}{\lambda} \right)$$

We've dealt with these kinds of problems before, in previous class materials. We know to wrap the constant into the function being differentiated. Let us perform a substitution: $y(t) = x(t) + \frac{u_0}{\lambda}$. Note that $\frac{d}{dt} y(t) = \frac{d}{dt} x(t)$, because the second term on the right hand side is a constant.
The differential equation becomes:

$$\frac{d}{dt} y(t) = \lambda y(t)$$

which we know has the solution:

$$y(t) = A e^{\lambda t}$$

Substituting $x(t)$ back into the equation,

$$x(t) = A e^{\lambda t} - \frac{u_0}{\lambda}$$

We find the constant A by using the initial condition: $x_d[0] = A - \frac{u_0}{\lambda}$. This implies $A = x_d[0] + \frac{u_0}{\lambda}$.
Substituting back in for A gives:

$$x(t) = A e^{\lambda t} - \frac{u_0}{\lambda} = e^{\lambda t} \left(x_d[0] + \frac{u_0}{\lambda} \right) - \frac{u_0}{\lambda}.$$

We can also evaluate the analytic answer:

$$\begin{aligned} x(t) &= e^{\lambda t} x_d[0] + e^{\lambda t} \int_0^t e^{-\lambda \tau} u_c(\tau) d\tau = e^{\lambda t} x_d[0] + e^{\lambda t} \int_0^t e^{-\lambda \tau} u_0 d\tau \\ &= e^{\lambda t} x_d[0] + u_0 e^{\lambda t} \int_0^t e^{-\lambda \tau} d\tau = e^{\lambda t} x_d[0] + u_0 e^{\lambda t} \int_0^t e^{-\lambda \tau} d\tau \\ &= e^{\lambda t} x_d[0] + u_0 e^{\lambda t} \left[-\frac{e^{-\lambda \tau}}{\lambda} \right]_0^t = e^{\lambda t} x_d[0] + \frac{u_0}{\lambda} - \frac{u_0}{\lambda} \\ &= e^{\lambda t} \left(x_d[0] + \frac{u_0}{\lambda} \right) - \frac{u_0}{\lambda} \end{aligned}$$

We see that the analytic solution works for both $u(t) = 0$ and constant $u(t) = u_0$.

(i) Verify the analytic solution found in (g) by plugging it back into the differential equation.

Answer: Plugging in our expression for $x(t)$,

$$\begin{aligned}
\frac{d}{dt}x(t) &= \frac{d}{dt}\left(e^{\lambda t}x_d[0] + e^{\lambda t}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau\right) \\
&= \frac{d}{dt}e^{\lambda t}x_d[0] + \frac{d}{dt}e^{\lambda t}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau \\
&= \left(\frac{d}{dt}e^{\lambda t}\right)x_d[0] + \left(\frac{d}{dt}e^{\lambda t}\right)\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau + e^{\lambda t}\left(\frac{d}{dt}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau\right) \\
&= \lambda e^{\lambda t}x_d[0] + \lambda e^{\lambda t}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau + e^{\lambda t}\left(\frac{d}{dt}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau\right) \\
&= \lambda\left(e^{\lambda t}x_d[0] + e^{\lambda t}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau\right) + e^{\lambda t}\left(\frac{d}{dt}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau\right) \\
&= \lambda x(t) + e^{\lambda t}\left(\frac{d}{dt}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau\right)
\end{aligned}$$

By the second fundamental theorem of calculus,

$$\frac{d}{dt}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau = e^{-\lambda t}u_c(t),$$

so plugging back into the expression above,

$$\begin{aligned}
\frac{d}{dt}x(t) &= \lambda x(t) + e^{\lambda t}\left(\frac{d}{dt}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau\right) \\
&= \lambda x(t) + e^{\lambda t}\left(e^{-\lambda t}u_c(t)\right) \\
&= \lambda x(t) + u_c(t)
\end{aligned}$$

which is the correct differential equation.

(j) If input $u(t)$ is the linear sum of two other inputs $u(t) = c_1u_1(t) + c_2u_2(t)$, what does the solution look like? What does this mean?

Answer: Because integration is linear, the solution will be

$$\begin{aligned}
x(t) &= e^{\lambda t}x_d[0] + e^{\lambda t}\int_0^t e^{-\lambda\tau}u_c(\tau)d\tau \\
&= e^{\lambda t}x_d[0] + e^{\lambda t}\int_0^t e^{-\lambda\tau}(c_1u_1(\tau) + c_2u_2(\tau))d\tau \\
&= e^{\lambda t}x_d[0] + c_1e^{\lambda t}\int_0^t e^{-\lambda\tau}u_1(\tau)d\tau + c_2e^{\lambda t}\int_0^t e^{-\lambda\tau}u_2(\tau)d\tau
\end{aligned}$$

We see that each of the two individual inputs $u_1(t)$ and $u_2(t)$ has a separate effect on $x(t)$. This means that we can evaluate each of the input's effects on the system separately, and then add those effects together to get the overall effect. This is essentially the superposition principle.

This problem brings together many different concepts and uses a lot of notation. As such, it may be difficult to fully comprehend everything the first time. *It is perfectly fine* to go back and spend more time on the problem until you completely understand it. Being able to quickly analyze complex mathematical problems like this is part of the vaunted "mathematical maturity" that this class helps you foster. As the semester continues, you will find that these kinds of problems will seem progressively easier, both to understand quickly and to solve. But it won't happen without practice.

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