EECS 16B Spring 2021 — Designing Information Devices and Systems II — Discussion Worksheet

Discussion 2B

For this discussion, Note 1 is helpful.

1. RC Circuits

In this problem, we will be using differential equations to find the voltage across a capacitor over time in an RC circuit. We set up our problem by first defining four functions over time: $I(t)$ is the current at time t, $V(t)$ is the voltage across the circuit at time t, $V_R(t)$ is the voltage across the resistor at time t, and $V_C(t)$ is the voltage across the capacitor at time t.

Recall from 16A that the voltage across a resistor is defined as $V_R = RI_R$ where I_R is the current across the resistor. Also, recall that the voltage across a capacitor is defined as $V_C = \frac{Q}{C}$ where Q is the charge across the capacitor.

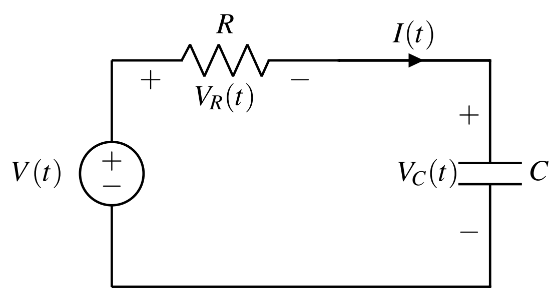

Figure 1: Example Circuit

(a) First, find an equation that relates the current across the capacitor $I(t)$ with the voltage across the capacitor $V_C(t)$.

Answer: We start from the Q-V relationship of the capacitor:

$$Q(t) = CV_C(t).$$

Differentiating $V_C(t) = \frac{Q(t)}{C}$ in terms of t, we get

$$\frac{dV_C(t)}{dt} = \frac{dQ(t)}{dt}\frac{1}{C}.$$

By definition, the change in charge is the current across the capacitor, so

$$\frac{d}{dt}V_C(t) = I(t)\frac{1}{C}$$

(b) Write a system of equations that relates the functions $I(t)$, $V_C(t)$, and $V(t)$.

Answer: From KVL,

$$V(t) - V_R(t) - V_C(t) = 0$$

$$RI(t) + V_C(t) = V(t) \quad (1)$$

(c) So far, we have three unknown functions and only one equation, but we can remove $I(t)$ from the equation using what we found in part (a). Rewrite the previous equation in part (b) in the form of a differential equation.

Answer:

From part (a), we have

$$I(t) = \frac{dV_C(t)}{dt}C$$

Substituting this into Equation 1 gives us

$$RC\frac{dV_C(t)}{dt} + V_C(t) = V(t)$$

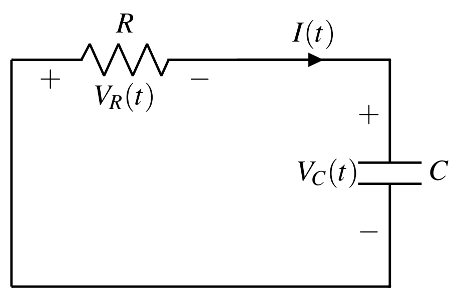

Figure 2: Circuit for part (d)

(d) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Answer:

Because $V(t) = 0$, our differential equation simplifies to

$$RC\frac{dV_C(t)}{dt} + V_C(t) = 0$$

Doing some algebraic manipulations gives us

$$\frac{dV_C(t)}{dt} = -\frac{1}{RC}V_C(t)$$

This equation tells us that we are looking for some function $V_C(t)$ such that when we take its derivative, we get the same function $V_C(t)$ multiplied by a scalar $-\frac{1}{RC}$. Because the derivative is equal to a scalar times itself, we think that the solution $V_C(t)$ will probably be of the form Ae^{bt}, where A and b are both constants.

Following the lecture, we first solve for A using the initial condition. Indeed, we have

$$\begin{aligned} V_{DD} &= V_C(0) \\ &= Ae^{b(0)} \\ &= A \end{aligned}$$

so that $V_C(t) = V_{DD}e^{bt}$. The last thing to do is to find b. We have already used the initial condition, so we must be able to find b from the differential equation. Plugging in for $V_C(t)$ in the differential equation, we have

$$\begin{aligned}
\frac{d}{dt}V_C(t) &= -\frac{1}{RC}V_C(t) \\
\frac{d}{dt}V_{DD}e^{bt} &= bV_{DD}e^{bt} = bV_C(t) \\
&= -\frac{1}{RC}V_C(t) \\
b &= -\frac{1}{RC}.
\end{aligned}$$

In this case we see that $b = -\frac{1}{RC}$ and our solution is

$$V_C(t) = V_{DD}e^{-\frac{1}{RC}t}$$

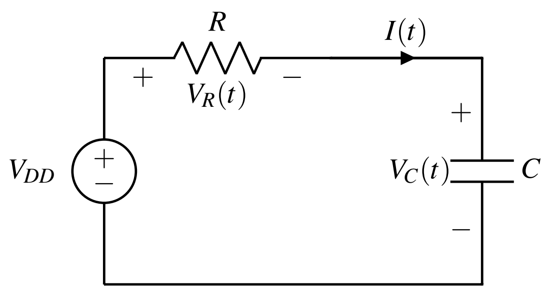

Figure 3: Circuit for part (e)

(e) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit. Solve the differential equation for $V_C(t)$ for $t \geq 0$.

Answer:

Substituting $V(t) = V_{DD}$ into our solution from part (c):

$$RC\frac{dV_C(t)}{dt} + V_C(t) = V_{DD}$$

We want to arrange this equation to be in a form that we know how to solve:

$$\frac{d}{dt}V_C(t) = \frac{V_{DD} - V_C(t)}{RC}$$

This is not quite the form we have seen before, as the term on the right is not equal to the term being differentiated. In general, we don't like constants in the differential equation. The derivative of a constant is zero, so we can wrap the constant into the function being differentiated using a change of variables, like so. Let's define a new variable $\tilde{V}_C(t) = V_C(t) - V_{DD}$. Note that $\frac{d\tilde{V}_C(t)}{dt} = \frac{dV_C(t)}{dt}$, and that $\tilde{V}_C(0) = V_C(0) - V_{DD} = -V_{DD}$. We can substitute these into our differential equation and obtain

$$\begin{aligned}
RC\frac{dV_C(t)}{dt} + V_C(t) - V_{DD} &= 0 \\
RC\frac{d\tilde{V}_C(t)}{dt} + \tilde{V}_C(t) &= 0 \\
\frac{d\tilde{V}_C(t)}{dt} &= -\frac{1}{RC}\tilde{V}_C(t)
\end{aligned}$$

In this equation, we have now removed V_{DD} from the left hand because of how we defined $\tilde{V}_C(t)$. And so we get back almost the same differential equation as in the previous part, this time for $\tilde{V}_C(t)$, with the only difference being that the initial condition changed! And so we can use that solution to get

$$\tilde{V}_C(t) = \tilde{V}_C(0)e^{-\frac{t}{RC}} = -V_{DD}e^{-\frac{t}{RC}}.$$

Finally, we need the solution in terms of $V_C(t)$ and not $\tilde{V}_C(t)$, so we back-substitute:

$$\begin{aligned} V_C(t) &= V_{DD} + \tilde{V}_C(t) \\ &= V_{DD} - V_{DD}e^{-\frac{t}{RC}} \\ &= V_{DD}(1 - e^{-\frac{t}{RC}}). \end{aligned}$$