1 KVL/KCL Review

Kirchhoff’s Circuit Laws are two important laws used for analyzing circuits. Kirchhoff’s Current Law (KCL) says that the sum of all currents entering a node must equal 0. For example, in Figure 1, the sum of all currents entering node 1 is $I_1 - I_2 - I_3 = 0$. Assuming that $I_1$ and $I_3$ are known, we can easily obtain a solvable equation for $V_x$ by applying Ohm’s law: $I_1 - \frac{V_x}{R_1} - I_3 = 0$.

Figure 1: KCL Circuit

Kirchhoff’s Voltage Law (KVL) states that the sum of all voltages in a circuit loop must equal 0. To apply KVL to the circuit shown in Figure 2, we can add up voltages in the loop in the counterclockwise direction, which yields $-V_1 + V_x + V_y = 0$. Using the relationships $V_x = i \cdot R_1$ and $i = I_1$, we can solve for all unknowns in this circuit. You can use these two laws to solve any circuit that is planar and linear.

Figure 2: KVL Circuit

If you would like to review these concepts more in-depth, you can check out the EECS16A Fall 2020 course notes.

2 Op-amp Review

Figure 3 shows the equivalent model of an op-amp. It is important to note that this is the general model of an op-amp, so our op-amp golden rules cannot be applied to this unless certain conditions are met.
Conditions Required for the Golden Rules:

(a) $R_{in} \to \infty$

(b) $R_{out} \to 0$

(c) $A \to \infty$

(d) The op-amp must be operated in negative feedback.

When conditions (a)-(c) are met, the op-amp is considered ideal. Figure 4 shows an ideal op-amp in negative feedback, which can be analyzed using the Golden Rules.

Golden Rules of ideal op-amps in negative feedback:

(a) No current can flow into the input terminals ($I_- = 0$ and $I_+ = 0$). (This property follows from condition (a) and does not require negative feedback.)

(b) The (+) and (−) terminals are at the same voltage ($V_+ = V_-)$.

If you would like to review these concepts more in-depth, you can check out op-amp introduction and op-amp negative feedback from the EECS16A course notes.
1. **KVL/KCL Review**

Use Kirchhoff’s Laws on the circuit below to find $V_x$ in terms of $V_{in}, R_1, R_2, R_3$.

![Example Circuit](image)

Figure 5: Example Circuit

(a) What is $V_x$?

**Answer:**

![Diagram with currents](image)

Figure 6

Applying KCL to the node at $V_x$, we get

$$
\frac{V_{in} - V_x}{R_1} - \frac{V_x - 0}{R_2} - \frac{V_x - 0}{R_3} = 0
$$

Solving this equation for $V_x$ yields

$$
V_x = V_{in} \frac{R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}
$$

(b) As $R_3 \to \infty$, what is $V_x$? What is the name we used for this type of circuit?

**Answer:**
As $R_3 \to \infty$, the $R_1R_2$ term on the denominator will become insignificant, simplifying our expression.

$$\lim_{R_3 \to \infty} V_x = \lim_{R_3 \to \infty} \frac{R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

$$= V_{in} \frac{R_2R_3}{R_1R_3 + R_2R_3}$$

$$= V_{in} \frac{(R_2)R_3}{(R_1 + R_2)R_3}$$

$$= V_{in} \frac{R_2}{R_1 + R_2}$$

When $R_3 \to \infty$, it effectively becomes an open wire, which makes the rest of the circuit a voltage divider, or resistive divider.

2. Op-Amp Summer

Consider the following circuit (assume the op-amp is ideal):

![Figure 7: Op-amp Summer](image)

What is the output $V_o$ in terms of $V_1$ and $V_2$? You may assume that $R_1$, $R_2$, and $R_g$ are known.

**Answer:**

![Figure 8](image)

Let $I_-$ be the current flowing into the (-) terminal of the op-amp.
\[ I_{R_g} + I_+ = \frac{V_1}{R_1} + \frac{V_2}{R_2} = I_{R_g} \]

\[ V_o = -R_g I_{R_g} \]

\[ V_o = -\left( \frac{R_g}{R_1} \cdot V_1 + \frac{R_g}{R_2} \cdot V_2 \right) \]
3. Current Sources And Capacitors (The following problem has been adapted from EECS16A Fall 20 Disc 9A.)

Recall charge has units of Coulombs (C), and capacitance is measured in Farads (F) = \frac{\text{Coulomb}}{\text{Volt}}.

It may also help to note metric prefix examples: 3µF = 3 \times 10^{-6} F.

Given the circuit below, find an expression for \( v_{\text{out}}(t) \) in terms of \( I_s, C, V_0, \) and \( t \), where \( V_0 \) is the initial voltage across the capacitor at \( t = 0 \).

Then plot the function \( v_{\text{out}}(t) \) over time on the graph below for the following conditions detailed below.

Use the values \( I_s = 1 \text{mA} \) and \( C = 2 \mu\text{F} \).

(a) Capacitor is initially uncharged \( V_0 = 0 \) at \( t = 0 \).
(b) Capacitor has been charged with \( V_0 = +1.5 \text{V} \) at \( t = 0 \).
(c) Practice: Swap this capacitor for one with half the capacitance \( C = 1 \mu\text{F} \), which is initially uncharged \( V_0 = 0 \) at \( t = 0 \).

HINT: Recall the calculus identity \( \int_a^b f'(x)dx = f(b) - f(a) \), where \( f'(x) = \frac{df}{dx} \).

Answer:
The key here is to exploit the capacitor equation by taking its time-derivative

\[ Q = C V_{\text{out}} \quad \longrightarrow \quad \frac{dQ}{dt} \equiv I_s = C \frac{dV_{\text{out}}}{dt} \]
From here we can rearrange and show that
\[ \frac{dV_{out}}{dt} = \frac{I_s}{C} \]

Thus the voltage has a constant slope!
Our solution is
\[ V_{out}(t) = V_0 + \left( \frac{I_s}{C} \right) \cdot t \]

To be more mathematically formal, we are solving a differential equation that happens to return a linear function for \( v_{out}(t) \):
\[ \frac{dV_{out}}{dt} = \frac{I_s}{C} \implies \int_0^t \frac{dV_{out}}{dt} \, dt \equiv V_{out}(t) - V_{out}(0) = \int_0^t \frac{I_s}{C} \, dt \equiv \frac{I_s}{C} \int_0^t 1 \, dt \equiv \frac{I_s}{C} \cdot t \]

Thus we arrive at the same statement as seen earlier \( V_{out}(t) = V_{out}(0) + \left( \frac{I_s}{C} \right) \cdot t \).

From this stage we can compute the slope of \( V_{out}(t) \) for parts (a) and (b) along with the slope for (c), which should be twice as large.

\[ \frac{I_s}{C} = \frac{1 \text{mA}}{2 \mu F} = \frac{1000 \mu \text{C}}{2 \mu \text{F}} = 500 \frac{\text{V}}{\text{s}} = \left( \frac{1}{2} \right) \frac{\text{V}}{\text{ms}} \]

For part (c):

\[ \frac{I_s}{C} = \frac{1 \text{mA}}{1 \mu F} = \frac{1000 \mu \text{C}}{1 \mu \text{F}} = 1000 \frac{\text{V}}{\text{s}} = 1 \frac{\text{V}}{\text{ms}} \]

When plotting, make sure to recall (a) and (c) start at the origin, while (b) has initially charged plates by \( V_0 = 1.5 \text{V} \). Results are shown below
4. Linear Algebra Review

For the following matrices, find the following properties:

i. What is the column space of the matrix?
ii. What is the null space of the matrix?
iii. What are the eigenvalues and corresponding eigenspaces for the matrix?

(a) \[
\begin{bmatrix}
2 & 4 \\
0 & 3
\end{bmatrix}
\]

Answer:

i. \( \mathbb{R}^2 \)

ii. 0

iii. \( \lambda_1 = 2, \ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

\( \lambda_2 = 3, \ v_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \)

(b) \[
\begin{bmatrix}
1 & -2 \\
2 & -4
\end{bmatrix}
\]

Answer:

i. span \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)

ii. span \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \)

iii. \( \lambda_1 = -3, \ v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)

\( \lambda_2 = 0, \ v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \)

(c) \[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 0.5 & 0.5 \\
0 & 0.5 & 0.5
\end{bmatrix}
\]

For this matrix you are told that the eigenvalues are: 2, 1, and 0.

Answer:

i. span \( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \)

ii. span \( \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \)
iii. $\lambda_1 = 2, \nu_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 1, \nu_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = 0, \nu_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$