EECS16B
Designing Information Devices and Systems II

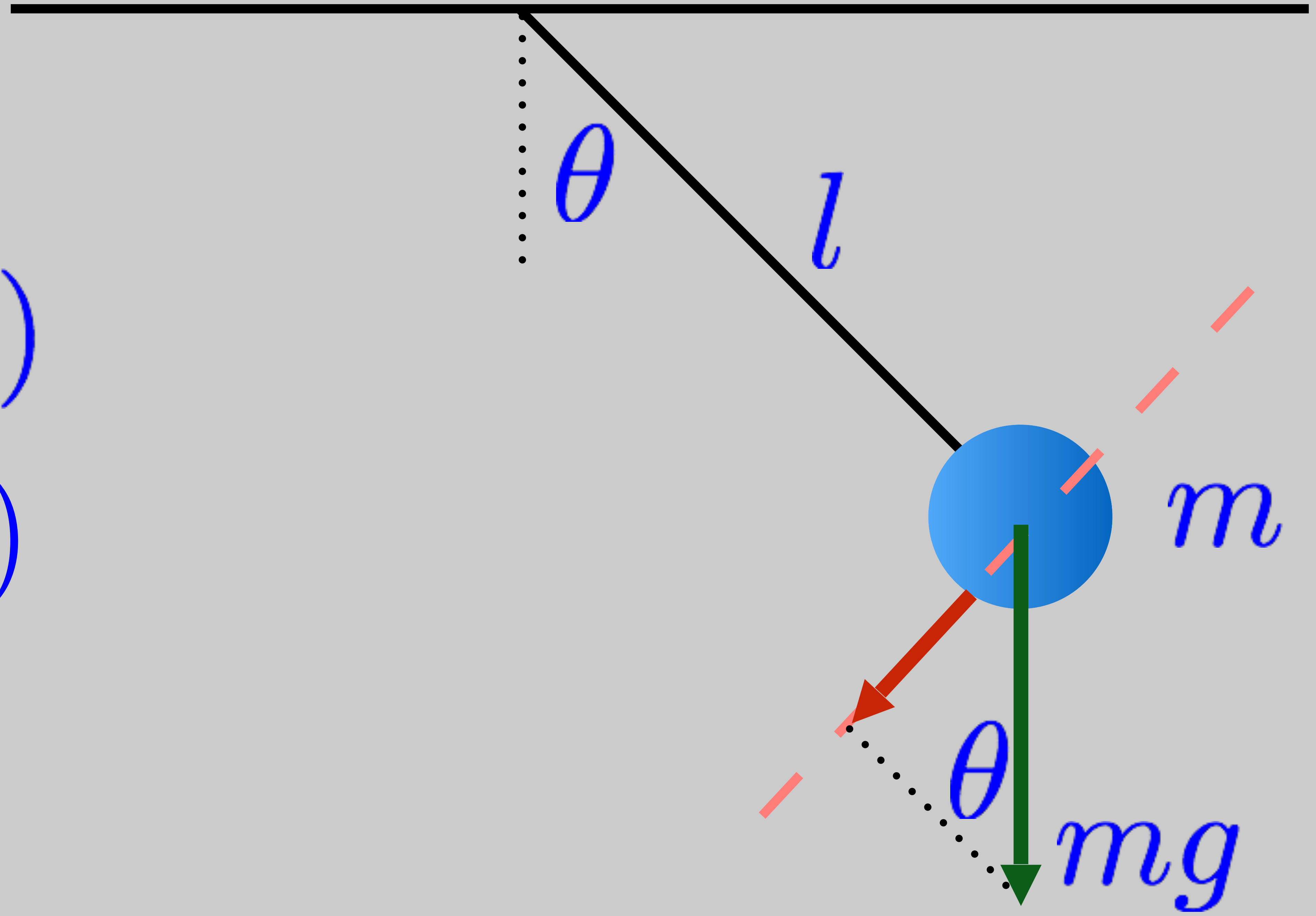
Lecture 15A
Linearization

Intro

- Last time
  - PCA
- Today
  - A bit on Linearization of non-linear systems
    - Taylor approximation
    - Gradient
    - Jacobian

https://www.youtube.com/watch?v=SPO9pVwoxVg

Linearization



State variables:

$$x_1(t) = \theta(t)$$
$$x_2(t) = \dot{\theta}(t)$$

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t)$$

$\approx x_1$

Linearization:

$$\frac{dx_2(t)}{dt} = -\frac{g}{l}x_1(t) - \frac{k}{m}x_2(t)$$

Linearization

$$\frac{dx_1(t)}{dt} = x_2(t)$$
$$\frac{dx_2(t)}{dt} = -\frac{g}{l}x_1(t) - \frac{k}{m}x_2(t)$$

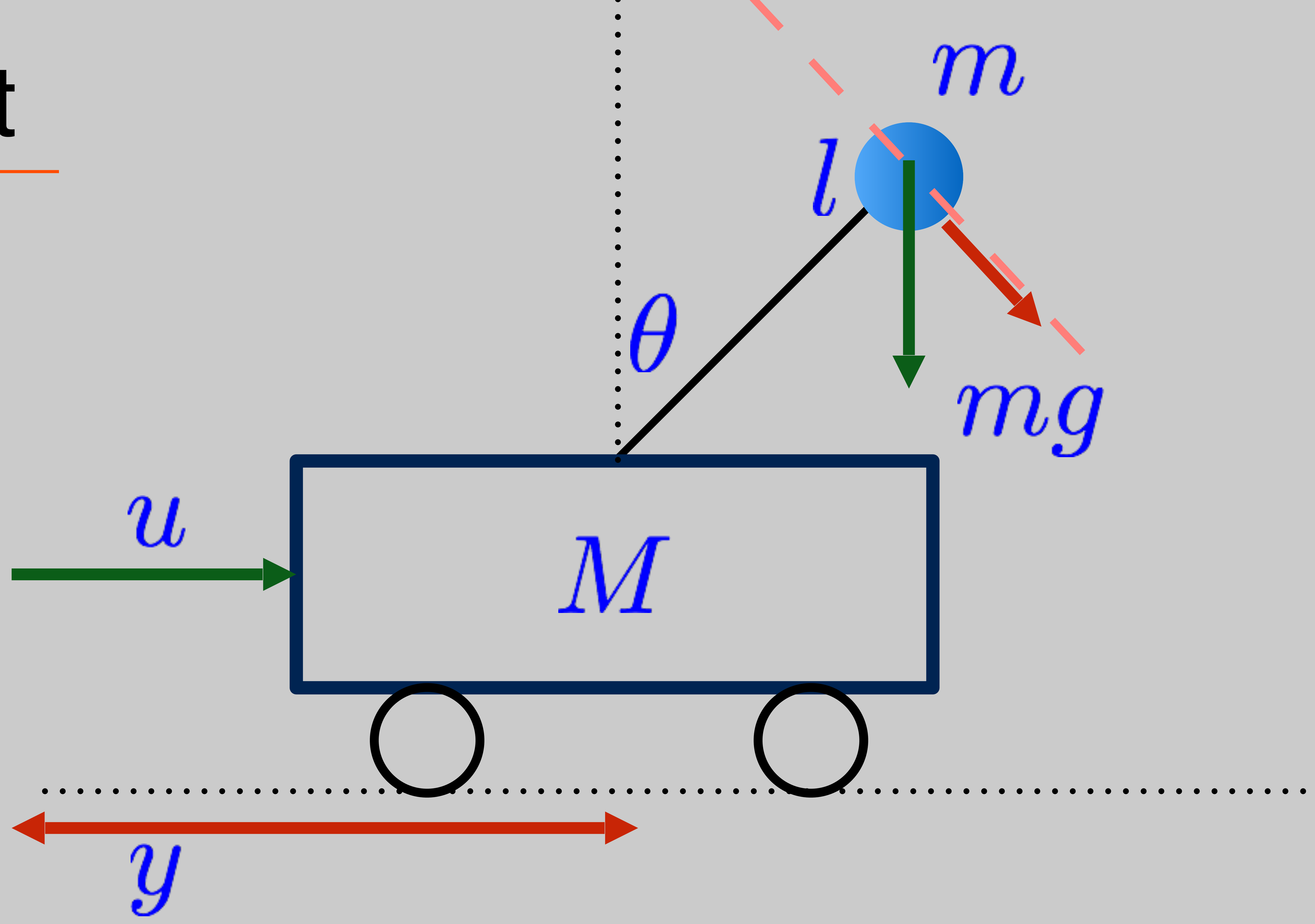
$$\Rightarrow \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$


DELL

Scary Example: Pole on a Cart

How many state variables?

How to systematically linearize?



$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 l \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{l(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 l \sin\theta\cos\theta + \frac{M+m}{m} g\sin\theta\right)$$

Linearization- teaching approach

Start with scalar

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

Continue to Vector to scalar functions

$$f : \mathbb{R}^N \rightarrow \mathbb{R}$$

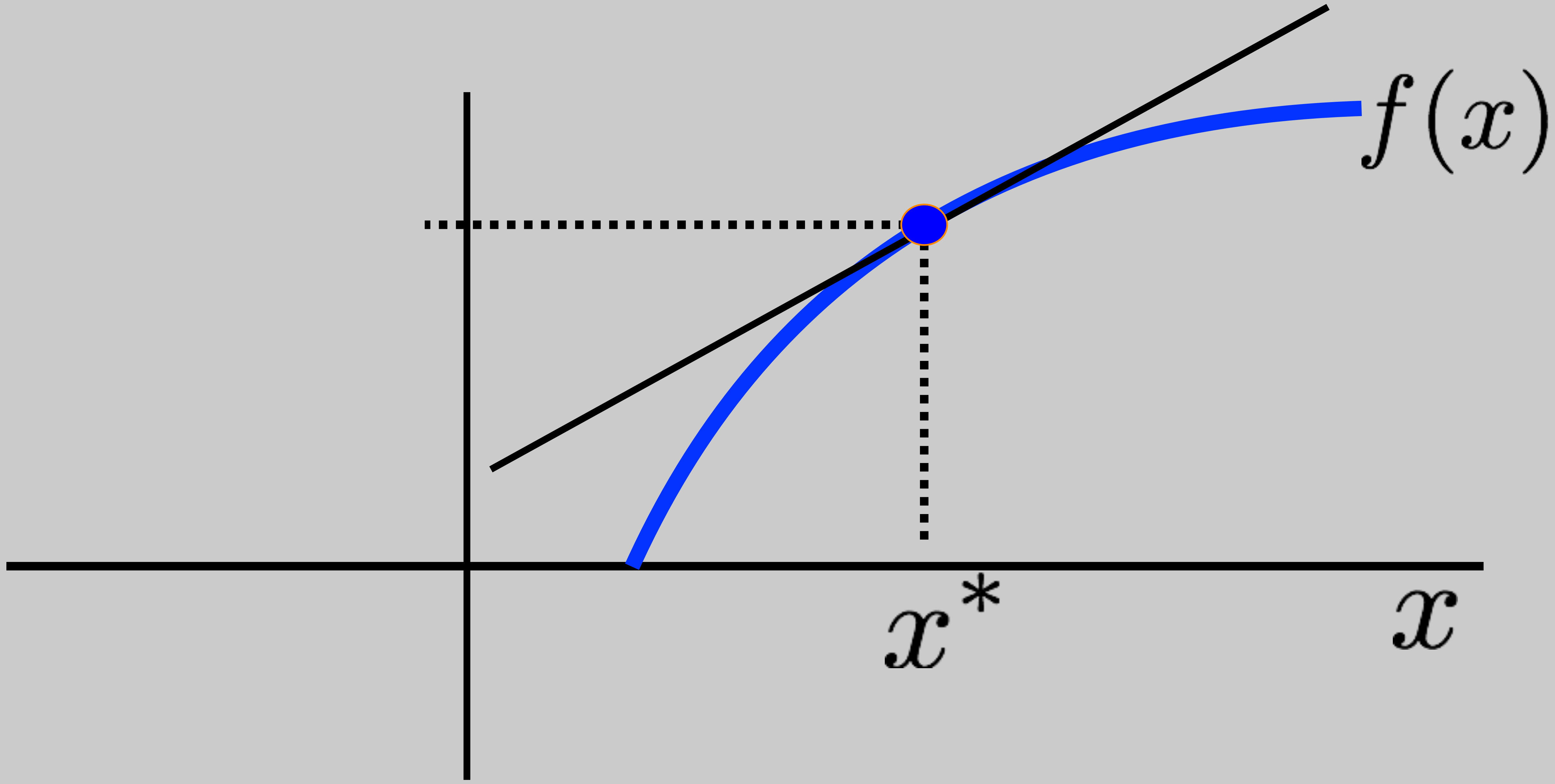
First for N=2 and slow derivation

Then – show math syntax sugar (gradient)

Generalize to $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ (Jacobian)

Taylor Approximation - scalar

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

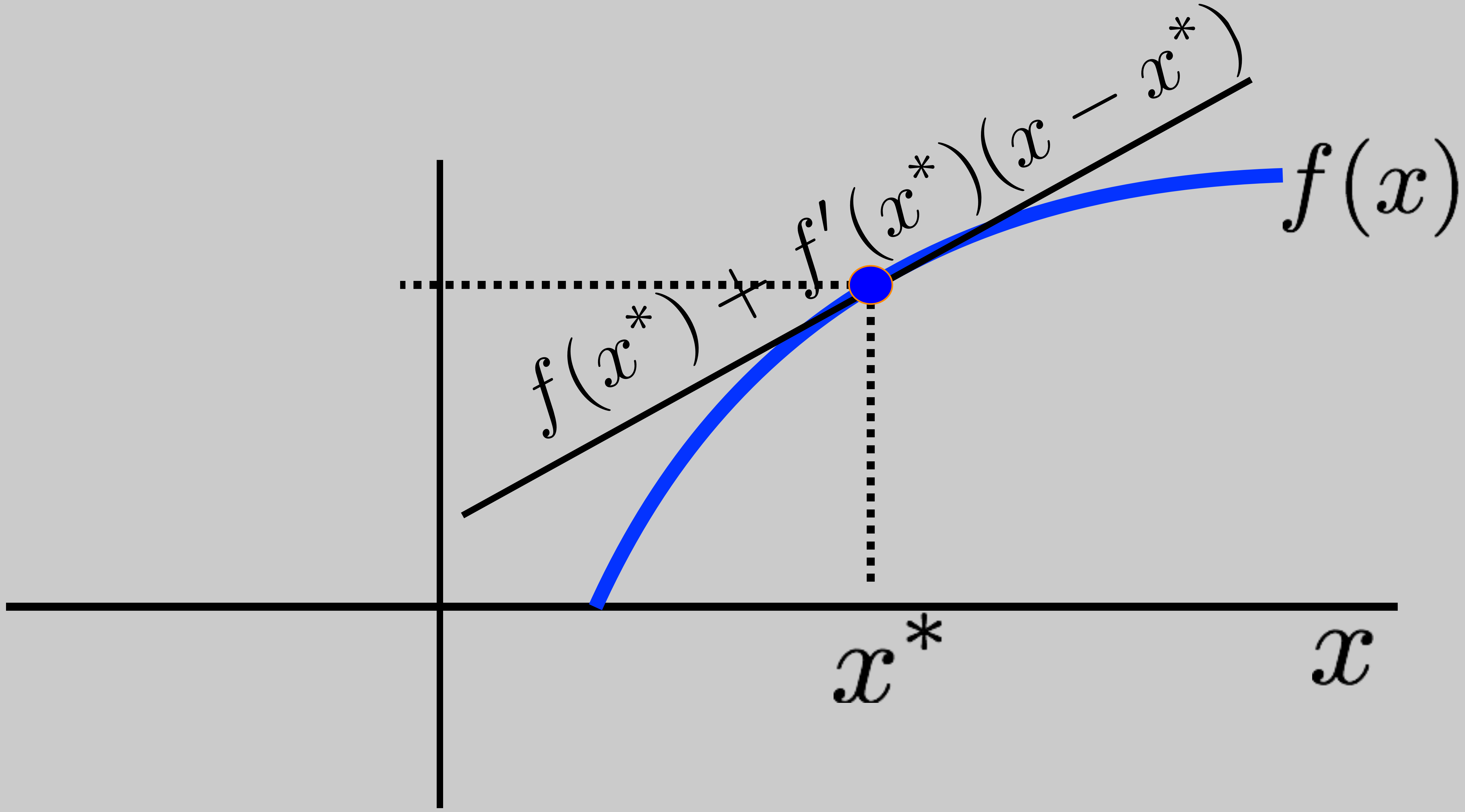


Taylor Approximation - scalar

$$f : \mathbb{R} \to \mathbb{R}$$

Taylor Approximation - scalar

$$f : \mathbb{R} \to \mathbb{R}$$



$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$\Rightarrow \sin(x) \approx \sin(x^*) + \cos(x^*)(x - x^*)$$

$$x^* = 0 \Big| \Rightarrow \sin(x) \approx \sin(0) + \cos(0)(x - 0)$$

$$\sin x \approx x$$

Taylor Approximation - scalar

$f : \mathbb{R} \rightarrow \mathbb{R}$

$$\Rightarrow \sin(x) \approx \sin(x^*) + \cos(x^*)(x - x^*)$$

Example:

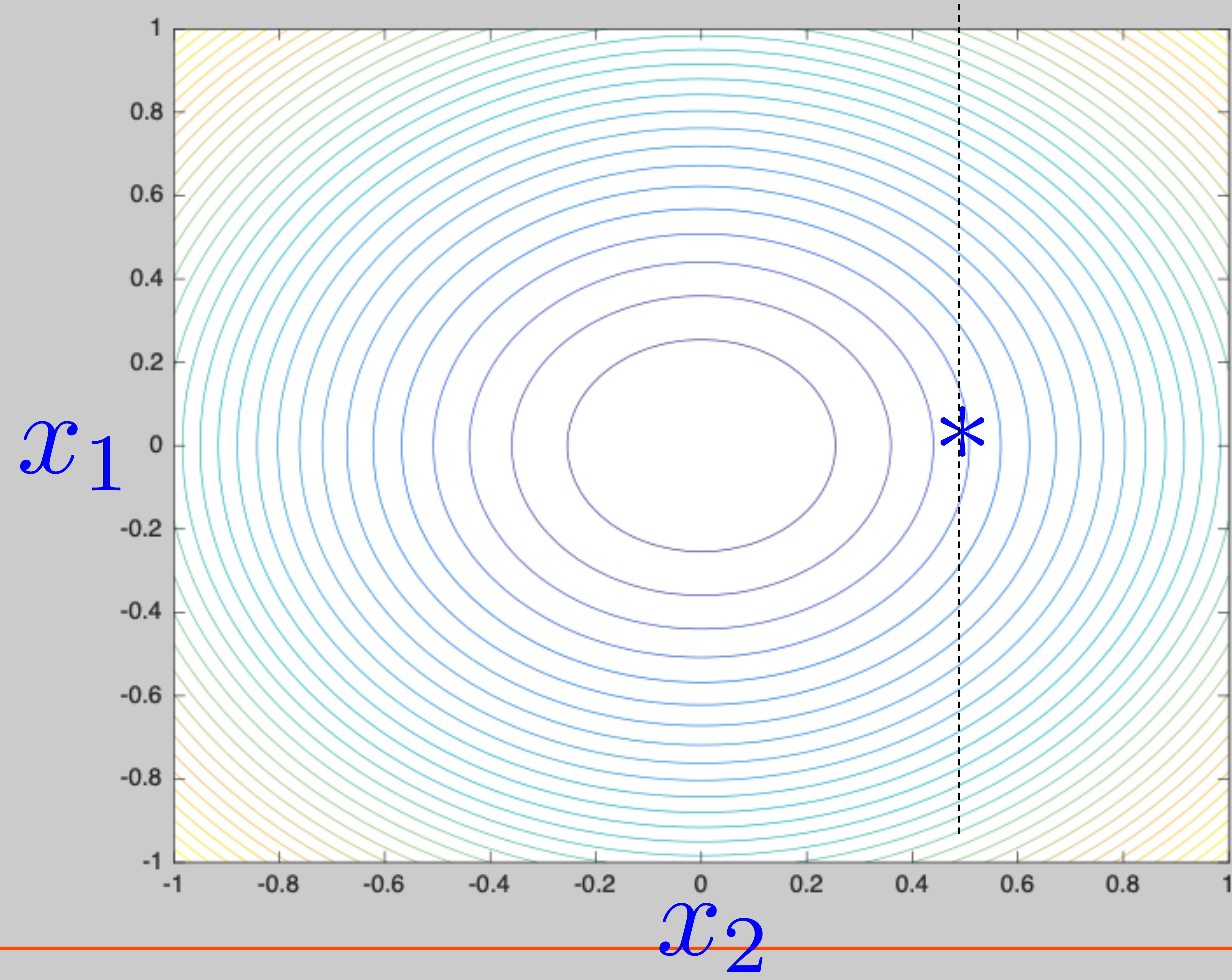
Taylor Approximation - vector

$$f : \mathbb{R}^N \rightarrow \mathbb{R}$$

Example: $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ $\quad f(\vec{x}) = ||\vec{x}||^2 = x_1^2 + x_2^2$

Let's look at

$$f(x_1, x_2 = x_2^*) = x_1^2 + {x_2^*}^2$$



Partial Derivative

Scalar "template"

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$f(x_1, x_2 = x_2^*) = x_1^2 + {x_2^*}^2$$

$$\frac{d}{dx_1} f(x_1, x_2^*) = \frac{d}{dx_1} x_1^2 + \frac{d}{dx_1} {x_2^*}^2 = 2x_1$$

$$\frac{\partial}{\partial x_1} f(x_1, x_2) = 2x_1$$

$$\frac{\partial}{\partial x_2} f(x_1, x_2) = 2x_2$$

Taylor Approximation - vector

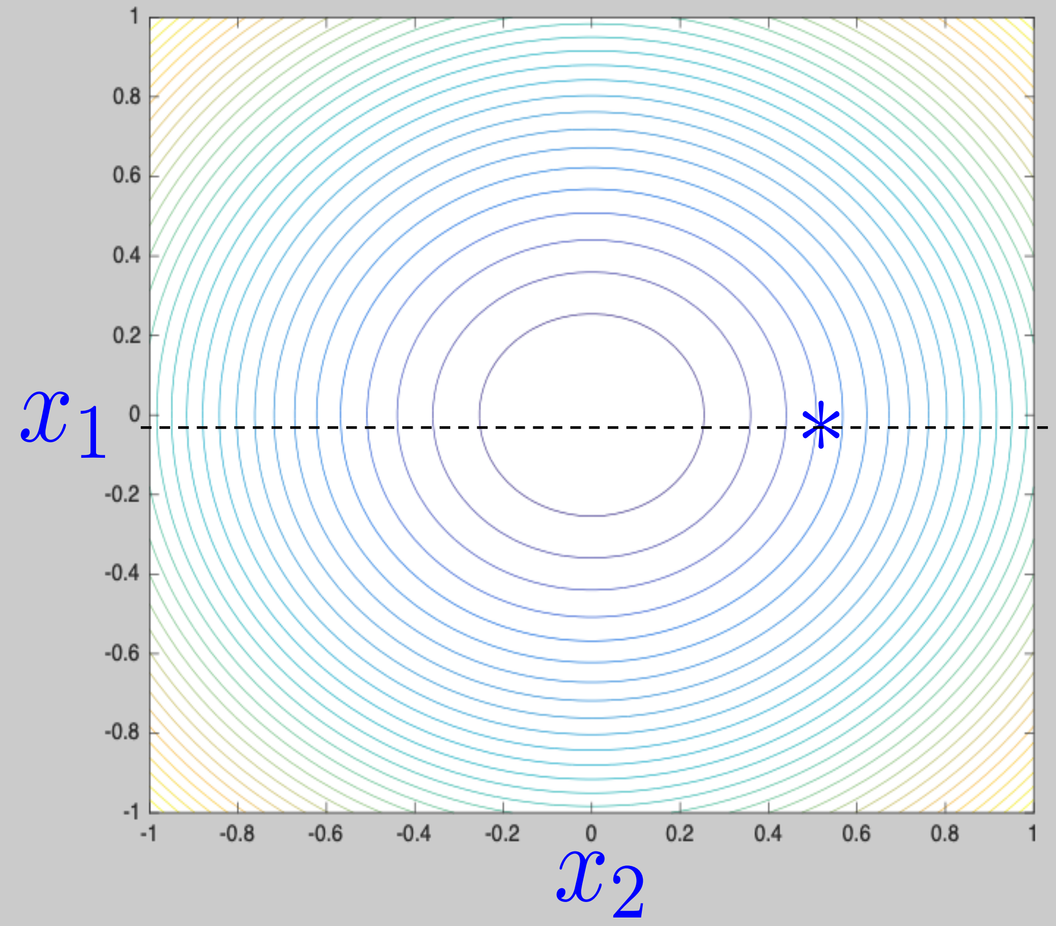
Scalar “template”

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$f(x_1, x_2 = x_2^*) = x_1^2 + {x_2^*}^2$$

$$\frac{\partial}{\partial x_1} f(x_1, x_2)\Big|_{x_1^*, x_2^*} = 2x_1^*$$

$$f(x_1, x_2^*) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_1^*(x_1 - x_1^*)$$



Similarly:

$$f(x_1^*, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_2^*(x_2 - x_2^*)$$

So,

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_1^*(x_1 - x_1^*) + 2x_2^*(x_2 - x_2^*)$$

Taylor Approximation - vector

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + 2x_1^*(x_1 - x_1^*) + 2x_2^*(x_2 - x_2^*)$$

Write in vector form:

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + \begin{bmatrix} 2x_1^* & 2x_2^* \end{bmatrix} (\vec{x} - \vec{x}^*)$$

Taylor Approximation - vector

Scalar "template"

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$$

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$f(x_1, x_2) \approx {x_1^*}^2 + {x_2^*}^2 + \begin{bmatrix} 2x_1^* & 2x_2^* \end{bmatrix} (\vec{x} - \vec{x}^*)$$

$$f(\vec{x}) \approx f(\vec{x}^*) + \begin{bmatrix} \frac{\partial}{\partial x_1} f(\vec{x}^*) & \frac{\partial}{\partial x_2} f(\vec{x}^*) \end{bmatrix} (\vec{x} - \vec{x}^*)$$

$$f(\vec{x}) \approx f(\vec{x}^*) + \nabla f(\vec{x}^*)(\vec{x} - \vec{x}^*)$$

Q: What are the dimensions of ∇f(x*) ? (gradient / Jacobian)

Taylor Approximation - vector

$$f : \mathbb{R}^N \to \mathbb{R}^N$$

$$\frac{d}{dt}\vec{x}(t) = f\left(\vec{x}(t)\right)$$

$$\underbrace{f(\vec{x})}_{Nx1} \approx \underbrace{f(\vec{x}^*)}_{Nx1} + \nabla f(\vec{x}^*)(\underbrace{\vec{x} - \vec{x}^*}_{Nx1})$$

Q: What are the dimensions of $\nabla f(\vec{x}^*)$? (Jacobian)

A: NxN ?

Taylor Approximation - vector

$$f : \mathbb{R}^N \to \mathbb{R}^N$$

$$f(\vec{x}) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\nabla f(\vec{x}) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

i,j th entry: $\frac{\partial f_i(x)}{\partial x_j}$

Taylor Approximation - vector

$$f : \mathbb{R}^N \to \mathbb{R}^N \qquad f(\vec{x}) = \begin{bmatrix} f_1(x_1, \cdots, x_N) \\ f_2(x_1, \cdots, x_N) \\ \vdots \\ f_N(x_1, \cdots, x_N) \end{bmatrix}$$

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ & \vdots & \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}$$

i,j th entry: $\frac{\partial f_i(x)}{\partial x_j}$

Taylor Approximation - vector

$$f : \mathbb{R}^N \to \mathbb{R}^N \qquad f(\vec{x}) = \begin{bmatrix} f_1(x_1, \cdots, x_N) \\ f_2(x_1, \cdots, x_N) \\ \vdots \\ f_N(x_1, \cdots, x_N) \end{bmatrix}$$

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i,j th entry: $\frac{\partial f_i(x)}{\partial x_j}$

Linearization of State-Space

Linearize around an equilibrium, a point s.t.:

$$f(\vec{x}^*) = 0$$

Q: why?

A: no change!

$$\frac{d}{dt}\vec{x} = f(\vec{x})$$

$$\approx f(\vec{x}^*) + \nabla f(\vec{x}^*)(\vec{x} - \vec{x}^*)$$

$f(\vec{x}^*) = 0$, $\quad \vec{x} - \vec{x}^* = \tilde{x}$

Which of the variables is a function of t?

write a state model for deviation!

Linearization of State-Space

$$\tilde{x} = \vec{x} - \vec{x}^*$$

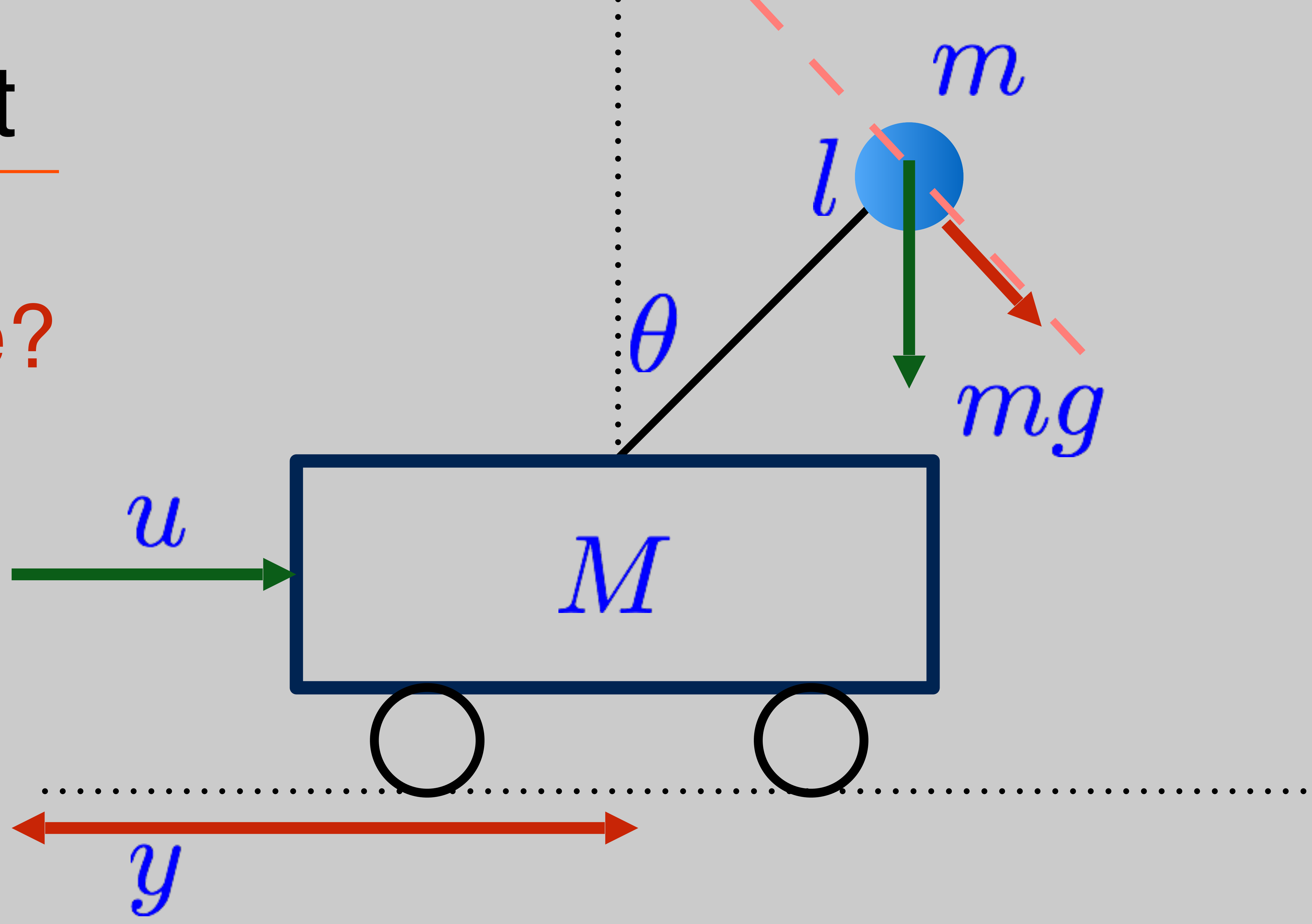
$$\frac{d}{dt}\tilde{x}(t) = \frac{d}{dt}\vec{x}(t) - \underbrace{\frac{d}{dt}\vec{x}^*}_{=0}$$

$$= f(\vec{x}(t)) \approx \underbrace{f(\vec{x}^*)}_{=0} + \nabla f(\vec{x}^*)\tilde{x}(t)$$

$$\frac{d}{dt}\tilde{x}(t) = [\nabla \underbrace{f(\vec{x}^*)}_{A}]\,\tilde{x}(t)$$

Scary Example: Pole on a Cart

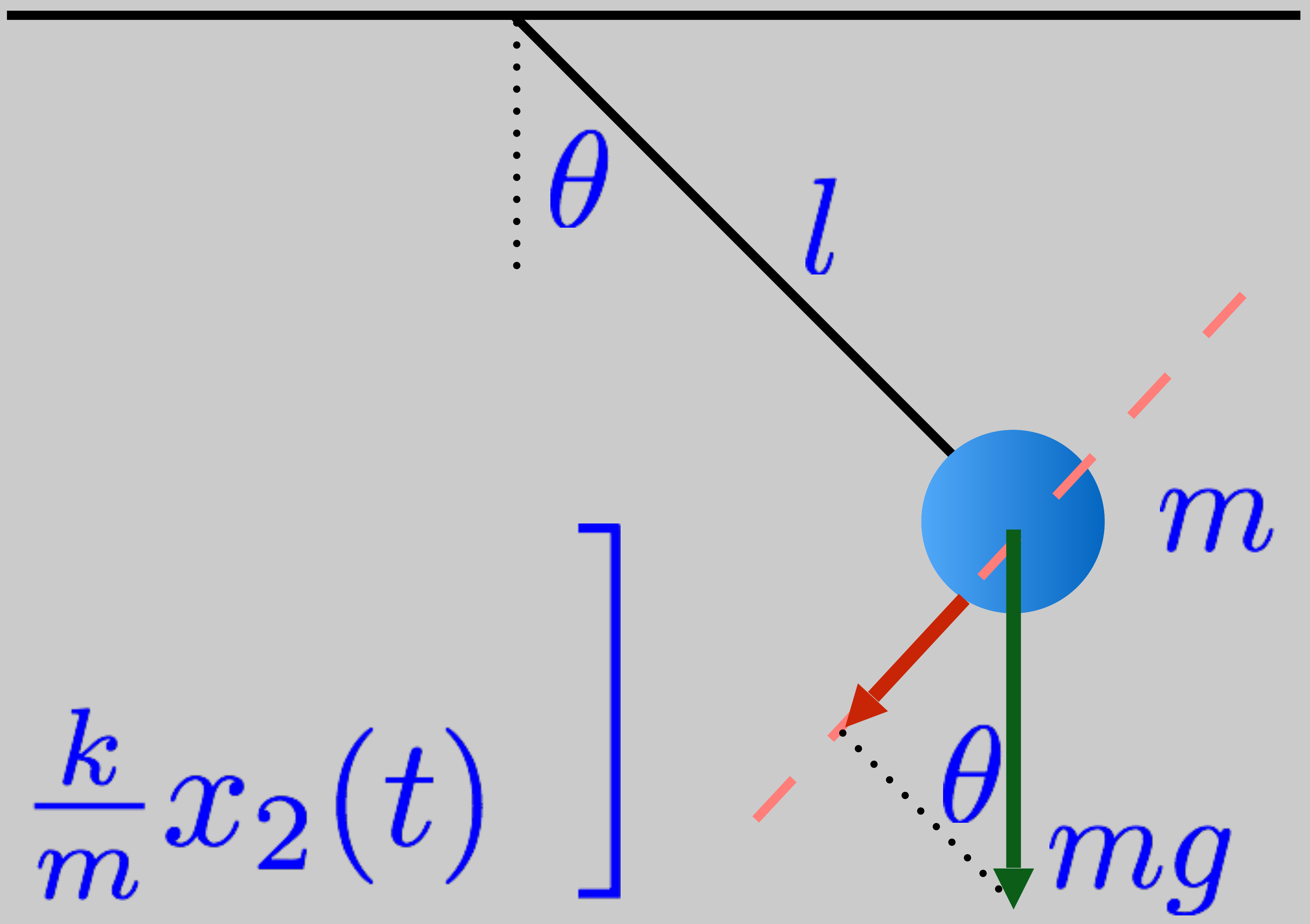
Q) Can you do it for this example?



$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 l \sin\theta - g\sin\theta\cos\theta\right)$$

$$\ddot{\theta} = \frac{1}{l(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 l \sin\theta\cos\theta + \frac{M+m}{m} g\sin\theta\right)$$

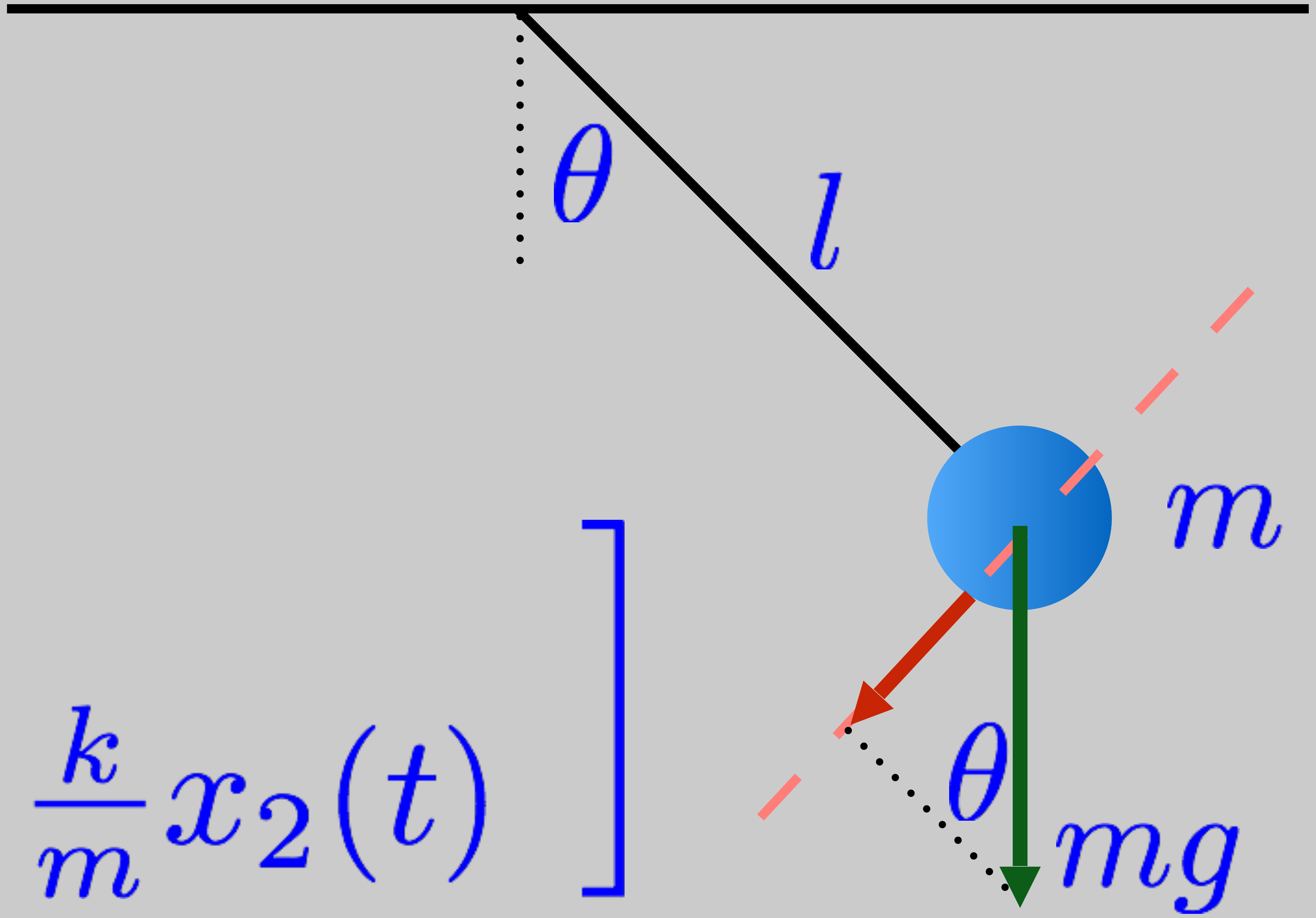
Back to the Pendulum



$$f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

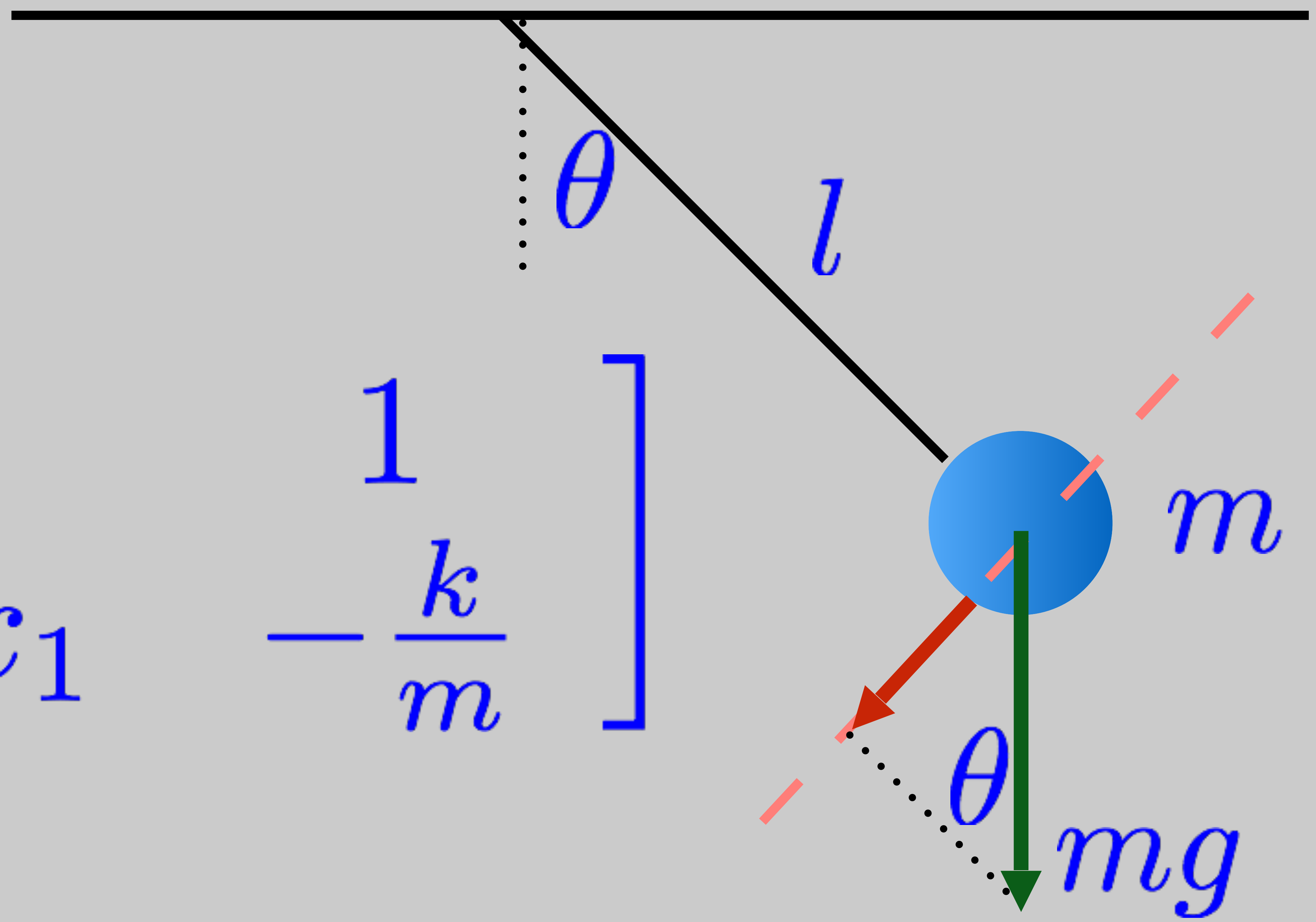
Back to the Pendulum



$$f(\vec{x}(t)) = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

Pendulum at Equilibrium



$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{m} \end{bmatrix}$$

$x_1^* = 0,\ x_2^* = 0$, Downward equilibrium

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

This is the same as small signal analysis!

$x_1^* = \pi,\ x_2^* = 0$, Upward equilibrium

$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

Eigen values?

Discrete Time

$$\vec{x}[n+1] = f\left(\vec{x}[n]\right)$$

$\vec{x} = \vec{x}^*$ is an equilibrium if:

$$f\left(\vec{x}^*\right) = \vec{x}^* \qquad \text{(for cont. } f\left(\vec{x}^*\right) = 0\text{)}$$

$$\tilde{x}[n] = \vec{x}[n] - \vec{x}^*$$

$$\tilde{x}[n+1] = \vec{x}[n+1] - \vec{x}^*$$

$$= f(\vec{x}[n]) - \vec{x}^*$$

$$\approx f(\vec{x}^*) + \overbrace{\nabla f\left(\vec{x}^*\right)}^{A} \tilde{x}[n] - \vec{x}^*$$

$$\tilde{x}[n+1] = A\vec{x}[n]$$

Summary

- Described linearization about an equilibrium point
  - Continuous time
  - Discrete time